5 Signals, Noise and Signal-to-Noise Ratio

5.1 Static Signal Conversion

A signal with a bitwidth of *NoB* (number of bits) can represent $NoL=2^{NOB}$ levels. Consequently we can say that the representation of *L-1* deltas (Δ) requires a number of

$$NoB = ceil(\,ld(NoL)\,) = ceil(\,log_B(NoL)/log_B(2)\,) = ceil(\,ln(NoL)/ln(2)\,)$$

bits, where function *ceil*(*x*) computes the next higher integer value and *ld* stands for *logarithmus dualis*, which is hardly on any computer but can be easily computed as

$$ld(x) = log_B(x)/log_B(2) = ln(x)/ln(2)\ .$$

with any positive base *B*. The accuracy of the measurement should be a half Δ (= least significant bit, LSB). Consequently, the integral non-linearity (*INL*) should be

$$INL \le 1 / 2^{NoB+1} \quad \Leftrightarrow \quad INL_{\%} \le 100\% / 2^{NoB+1}\ .$$

Example:
A DC voltmeter has a range of R=0…200 Δ. How many bits do we need for the ADC, what *INL* in % do we require for $INL \le ½Δ$?

200 Δ => NoL = 201 % Number of Levels:
NOB = ceil(ld(NoL)) = ceil(ln(201)/ln(2) = ceil(7.6) = 8

$INL_{\%} \le 100\% / 2^{8+1} = 0.2\%$

Exercise:
A DC voltmeter has a range of R=0…2000 Δ. How many bits do we need for the ADC, what $INL_{\%}$ do we require for $INL \le ½Δ$?

. . . .

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. . . .

Solution:
A DC voltmeter has a range of R=0…2000 Δ. How many bits do we need, what INL do we need when it should be $\le ½Δ$?
L=2001, NOB = ceil(ld(L)) bits = ceil(ln(2001)/ln(2)) bits = ceil(10.967) bits = 11 bits.
$INL \le 100\% / 2^{11+1} = 0.024\%$

5.2 Fundamentals on Handling Dynamic Signals

5.2.1 Signal Power and Effective / rms Amplitude

A signal is a physical representation of an information. It may come as voltage, current, power, temperature, displacement, as flag on an airport, as digital bit, etc. Except from some DC signals like temperature we typically handle waveforms like sound.

We distinguish between amplitude, power and effective amplitude, also termed root-mean-square (rms) value of a signal. Signal power is expressed as square of signal amplitude. Physically correct power is U^2/R and I^2R when U, I, and R represent voltage, current and resistor, respectively. But how to deal with other signals types like gas pressure, flags or digital signals? In signal processing the power of a signal is simply its squared amplitude. Average power is defined according to table 5.2.1.

Tab. 5.2.1: A signal's average value and average power

	Average or DC amplitude $\bar{x}$	Average signal power $\overline{x^2}$	Effective or *rms* amplitude x_{rms}
time continuous:	$x_{av} = \bar{x} = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\cdot dt$	$\overline{x^2} = \frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\cdot dt$	$x_{rms} = \sqrt{\overline{x^2}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\cdot dt}$
time discrete:	$x_{av} = \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$	$\overline{x^2} = \frac{1}{N}\sum_{i=1}^{N} x_i^2$	$x_{rms} = \sqrt{\overline{x^2}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} x_i^2}$
Multimeters:	RMS value of the	alternating part only:	$x_{rms\approx} = \sqrt{x_{rms\sim}^2 - x_{av}^2}$

The average value of a signal is termed its DC value, statistically represented as $\bar{x}$. With "signal power" we typically address the average power $\overline{x^2}$ of a signal power $x^2(t)$. The *effective* or *rms* amplitude of a signal is the square root of its average power.

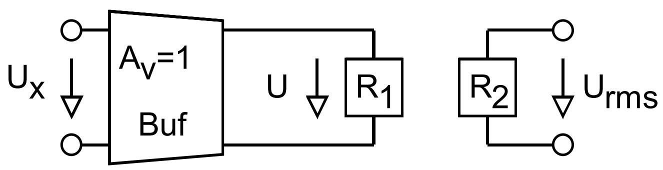


Fig. 5.2.1: The *effective* or *rms* value U_{rms} of a voltage $U(t)$ is the DC voltage, that causes the same heating of $R_2=R_1$ as $U(t)$.

Warning: The frequently seen notation $\bar{x}^2$ is not the average signal power but the square of its DC-value. Example: $x(t)=A\cdot\sin(\omega t)$ has a DC value of $\bar{x}=0$ and consequently $\bar{x}^2=0$, while its average power is $\overline{x^2} = A^2/2$.

5.2.2 Effective Values of Some Particular Waveforms

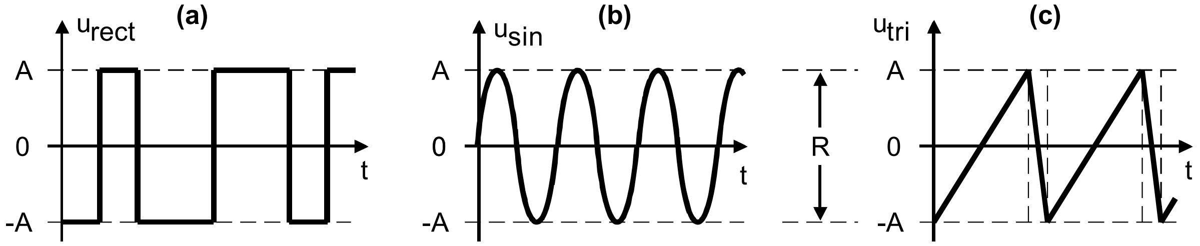


Fig. 5.2.2-1 : Particular waveforms: **(a)** rectangular, **(b)** sinusoidal, **(c)** triangular.

Fig. 5.2.2-1 shows (a) a rectangular, (b) a sinusoidal and (c) a triangular signal oscillating between the values A=R/2 and $-A$=-R/2 with range R=2A. Its total power for voltages at 1Ω and its effective voltages are given by

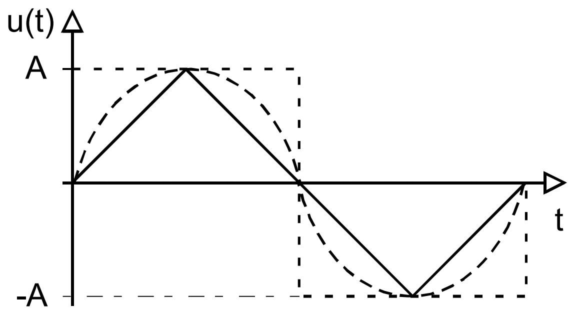
Rectangular: $$\overline{u_{rect}^2} = \frac{A^2}{1} = \frac{R^2}{4} \quad \leftrightarrow \quad u_{rect,eff} = \frac{A}{\sqrt{1}} = \frac{R}{2} \ , \qquad (5.\ )$$

Sinusoidal: $$\overline{u_{\sin}^2} = \frac{A^2}{2} = \frac{R^2}{8} \quad \leftrightarrow \quad u_{\sin,eff} = \frac{A}{\sqrt{2}} = \frac{R}{\sqrt{8}} \ , \qquad (5.\ )$$

Triangular: $$\overline{u_{tri}^2} = \frac{A^2}{3} = \frac{R^2}{12} \quad \leftrightarrow \quad u_{tri,eff} = \frac{A}{\sqrt{3}} = \frac{R}{\sqrt{12}} \ . \qquad (5.\ )$$

Different frequencies are uncorrelated, they add in power.

Fig. 5.2.2-2:
Area comparison of the three waveforms



<u>Exercises</u>

Exercise 1: Given is a rectangular waveform:
u_{rect}(t) =A while 0 ≤ t-nT ≤ T_H and u_{rect}(t) = -A while T_H ≤ t-nT ≤ T_H+T_L, n=0, 1, 2, 3, ...

Compute signal power: u^2_{rect}(t) = ...

Average signal power: u^2_{rms} = ...

Effective amplitude: u_{rms} = ...

Exercise 2: Given is a sinusoidal waveform: (Hint: $sin^2(x) = ½(1-cos(2x))$)
$u_{sin}(t) = A \cdot \sin(\omega t)$

Compute signal power: $u^2_{sin}(t)$ = ...

Average signal power: $u^2_{sin,rms}$ = ...

Effective amplitude: $u_{sin,rms}$ = ...

Exercise 3: Given is a triangular waveform:
$u_{tri}(t) = (A/T)\cdot t \quad \text{for} \quad 0 \le t-n\cdot T \le T, \quad n = 0,1,2,3,\ldots$

Compute signal power: $u^2_{tri}(t-nT)$ = ...

Average signal power: $u^2_{tri,rms}$ = ...

. . . .

Effective amplitude: $u_{tri,rms}$ = ...

What is the difference to power and rms-amplitude of $u_{tri}(t)$ if some triangles are positive and the others negative?
. . . .

Exercise 4:: Add an u_{offset} (f=0Hz) to DC-free, oscillating u_{osc} (f>0Hz):

. . . .

Solutions:

Exercise 1: Given is a rectangular waveform:
$u_{rect}(t) = A$ while $0 \le t-n\cdot T \le T_H$ and $u_{rect}(t) = -A$ while $T_H \le t-n\cdot T \le T_H+T_L$, n=0, 1, 2, 3, ...
Compute signal power: $u^2_{rect}(t)$ = **A²**
Average signal power: u^2_{rms} = **A²**
Effective amplitude: u_{rms} = **A**

Exercise 2: Given is a sinusoidal waveform:
$u_{sin}(t) = A \cdot \sin(\omega t)$
Compute signal power: $u^2_{sin}(t)$ = **$A^2 \cdot \sin^2(\omega t) = ½ A^2 \cdot (1-\cos(2\omega t))$**
Average signal power: $u^2_{sin,rms}$ = **½ A² as average over cos(x)=0.**
Effective amplitude: $u_{sin,rms}$ = **A/sqrt(2)**

Exercise 3: Given is a triangular waveform:
$u_{tri}(t) = (A/T)\cdot t \quad \text{for} \quad 0 \le t-n\cdot T \le T, \quad n = 0,1,2,3,\ldots$
Compute signal power: $u^2_{tri}(t-nT)$ = **$(A/T)^2 \cdot t^2 \cdot$**
Average signal power: $u^2_{tri,rms}$ = **$(1/T)\,[(A/T)^2 \cdot t^3/3]_0^T = (A/T)^2 \cdot T^3/3T = A^2/3$**
Effective amplitude: $u_{tri,rms}$ = **A/sqrt(3)**
What is the difference to power and rms-amplitude of $u_{tri}(t)$ if some triangles are positive and the others negative? **no difference**

Exercise 4: $\overline{u^2_{total}} = \overline{u^2_{offset}} + \overline{u^2_{osc}} \quad \rightarrow \quad u_{total,rms} = \sqrt{u^2_{offset} + u^2_{osc,rms}}$

5.2.3 Summation of Correlated and Uncorrelated Signals

- Correlated signals depend on each other
- Uncorrelated signals do not depend on each other

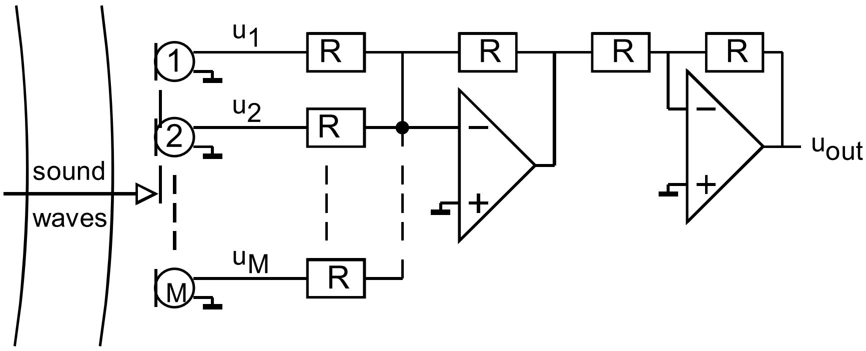
Correlated signals sum in amplitude: $y_{sum,corr} = x_1 + x_2 + x_3 + \ldots + x_N$ (5.)

Uncorrelated signals sum in power: $y_{sum,uncorr} = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots + x_N^2}$ (5.)

Different frequencies are always uncorrelated. (5.)

Exercise:
We have *N* identical microphones recording sound. The recorded sound waves are added optimally for amplification What is the improvement in SNR compared to a single microphone?

Fig. 5.x: *M* microphones receiving the same sound signal.



Solution to the exercise:
Sound waves are correlated an sum in amplitude: $U_{s,sum} = N \cdot U_s \Rightarrow P_s = N^2 \cdot U_s^2$.

The microphones noise in uncorrelated an sums in power: $P_{n,sum} = N \cdot U_n^2$.

SNR improves according to $SNR_{sum} = (N^2 \cdot U_s^2)/(N \cdot U_n^2) = N \cdot U_s^2/\cdot U_n^2$.
Consequently, the SNR improves by a factor N. This corresponds to factor sqrt(N) in voltages.

This is basically the first example of oversampling with oversampling ratio OSR=N.

5.2.4 Bel and Decibel

In honor of Graham Bell a factor 10 in signal power is termed a Bel, and 1 B = 10 dB, just as 1 m = 10 dm or 1 liter = 10 dl. As power corresponds to square of amplitude ($p=u^2/R=i^2\cdot R$) and $\log(x^2)=2\cdot\log(x)$ we get

$$\text{Signal-Ratio} = \log_{10}\frac{p_2}{p_1}B = 10\log_{10}\frac{p_2}{p_1}dB = 20\log_{10}\frac{u_2}{u_1}dB = 20\log_{10}\frac{i_2}{i_1}dB \quad . \tag{2.6}$$

where lg stands for $\log_{10}$.

Exercise 1: 10dB is what factor in signal power? 10dB is what factor in effective voltage?

. . . .

Exercise 2: A factor 2 in amplitude corresponds to one bit. Compute it in dB!

. . . .

Exercise 3: to what factor in amplitude and power do 3.01 dB correspond?

. . . .

Exercise 1:: 10dB is what factor in signal power? 10dB is what factor in effective voltage?
By definition 1 B = 10dB is a factor 10 in power → a factor sqrt(10)≈3.162 in voltage.

Exercise 2: A factor 2 in amplitude corresponds to one bit. Compute it in dB!
20dB·lg(2) ≈ 6.02 dB

Exercise 3: to what factor in amplitude and power do 3.01 dB correspond?
Amplitude: 10^(3.01dB/20dB)=sqrt(2), Power: 10^(3.01dB/10dB)= 2.

5.2.5 Signal Accuracy and Effective Number of Bits (*ENoB*)

This chapter is to give an intuitive introduction in A/D and D/A converter design and selection for engineers. We shall show that from theoretical considerations an *NoB* bit quantizer can obtain a maximum theoretical signal-to-noise ratio or signal-to-(noise+distorition) ratio (SINAD). While SNR is defined for any input waveform, SINAD assumes a maximum amplitude sinusoidal input wave. In this case SNR=SINAD.

$$\boldsymbol{SINAD_{dB}} = \lg\left(\frac{SignalPower}{NoisePower}\right)\cdot 10dB \le (\lg(2^{NoB}) + \lg(1.5))\ 10\text{dB} \approx \mathbf{(\boldsymbol{NoB}\cdot 6.02 + 1.76)\ dB}$$

or the effective number of bits (using *ENoB* = *NoB*) as

$$\boldsymbol{ENoB} = \frac{SINAD_{dB} - 1.76dB}{6.02dB} \text{ for triangular } e_q(\text{t}),\ \boldsymbol{ENoB} = (\text{SNR}_{\text{dB}} + 3.01\text{dB})/6.02\text{dB for rect. } e_q(\text{t}).$$

where *lg* stand for log_{10}. As a rule of thumb for nowadays ADCs compute

$$\boldsymbol{SINAD_{dB}} = \mathbf{(6\ \boldsymbol{ENoB} + 2)\ dB}$$

$$\boldsymbol{ENoB} \cong \frac{SINAD_{dB} - 2dB}{6dB}$$

Note that the 10 dB·*lg*(3/2)=1.76dB accounts for the different waveforms: While the reference signal is assumed to be sinusoidal, the quantization noise is assumed to have a triangular shape.

Exercises:

What is the maximum $SINAD_{dB}$ theoretically obtainable with a 16 bit ADC?

Rule of thumb: $\boldsymbol{SINAD_{dB}}$ **=** . . . .

Accurate: $\boldsymbol{SINAD_{dB}}$ **=** . . . .

In an advertisement a 16 bit ADC has a maximum *SINAD* of 93.5dB. What is its effective number of bits?

Rule of thumb: ***ENoB* =** . . . .

Accurate: ***ENoB* =** . . . .

Take data sheets of different vendors (e.g. Analog Devices, Burr Brown, Maxim, Linear technology, Texas Instruments,...) and check bit-width versus *SNR* for different ADCs.

Solutions:

Rule of thumb: $\mathrm{SINAD_{dBca}}$ = 16·6dB + 2dB = 98 dB
Accurate: $\mathrm{SINAD_{dB}}$ = 16·6.02 + 1.76)dB = 98.08dB
Rule of thumb: $\mathrm{ENoB_{ca}}$ = (93.5-2)dB / ·6dB = 15.25 bits
Accurate: ENoB = (93.5-1.76)dB·/ 6.02dB = 15.24 bits

5.2.6 Integration of Odd and Even f(x) in Symmetric Boundaries

A function is called

even	when	$f_{even}(x) = f_{even}(-x)$,	e.g. cos(x),
odd	when	$f_{odd}(x) = -f_{odd}(-x)$,	e.g. sin(x).

For integration in symmetric boundaries holds the rule

$$\int_{-B}^{B} f_{even}(x) \cdot dx = 2\int_{0}^{B} f_{even}(x) \cdot dx$$

$$\int_{-B}^{B} f_{odd}(x) \cdot dx = 0$$

Any function *f(x)* can be subdivided into an *odd* and an *even* part:

$f_{even}(x) = \frac{1}{2}(f(x) + f(-x))$, e.g. $\cos(x) = (e^{jx} + e^{-jx})/2$

$f_{odd}(x) = \frac{1}{2}(f(x) - f(-x))$, e.g. $\sin(x) = (e^{jx} - e^{-jx})/2j$.

Get back the original function by

$f(x) = f_{even}(x) + f_{odd}(x)$, e.g. $e^{jx} = \cos(x) + j \cdot \sin(x)$.

5.3 Budgeting Noise Sources

In the following, we write E_x^2 as abbreviation of $\overline{E_x^2} = E_{x,rms}^2$ with x standing for q, *alias*, *clkj*, *nonlin*, *T&H*, *others…*

From the (ADC and DAC) customer point of view, we distinguish 2 kinds of noise sources: “Internal” noise sources, that are specific to a particular device, and “external” noise sources of the device, so that we have an influence on them through the design. If a noise source is internal or not, depends on the particular conversion device. While quantization noise is always internal and depends on the number of bits (*NoB*), Sample&Hold noises depends on the device. Examples: *AD’s LTC2308* ADC provides no internal sampler, while *TI’s ADC10* within *MSP430* does, but we can do settings to control that sampler

We will use $\boldsymbol{E_{int,rms}}$ and $\boldsymbol{E_{ext,rms}}$, abbreviated with $\boldsymbol{E_{int}}$ and $\boldsymbol{E_{ext}}$, respectively:

- $\boldsymbol{E_{int}}$: Build-in noise voltage coming unavoidable with a particular (ADC or DAC) device.
- $\boldsymbol{E_{ext}}$: Noise voltage contributions that occur outside a considered (ADC or DAC) device.
- Total noise power: $E_{tot}^2 = E_{\text{int}}^2 + E_{ext}^2$

About the word “power”

- “Power” is physically measured in Watts, while we measure it here in squared amplitudes, e.g. V^2, A^2, For *SNR* computations the results are the same.
- True power computation would have to respect the DC component of a signal. A sinusoidal signal measured from 0…R would deliver an rms power of $R^2/8+R^2/4$, not $R^2/8$.

Some typical noises sources are

1.	$\boldsymbol{E_q}$	quantization noise
2.	$\boldsymbol{E_{nonlin}}$	noise due to built-in non-linearity
3.	$\boldsymbol{E_{switch}}$	noise from switching currents and/or voltages
4.	$\boldsymbol{E_{clkft}}$	clock feed-through: switching noise caused by digital clock signal
5.	$\boldsymbol{E_{clkj}}$	noise due to clock jitter
6.	$\boldsymbol{E_{thermal}}$	thermal Johnson noise (resistors have spectral noise power of $4kT\cdot B$)
7.	$\boldsymbol{E_{pink}}$	$1/f$ noise
8.	$\boldsymbol{E_{alias}}$	aliasing noise
9.	$\boldsymbol{E_{T\&H}}$	noise due to track-&-hold process
10.	$\boldsymbol{E_{current}}$	noise caused by current flow, e.g. through doped semiconductors or grain boundaries
11.	$\boldsymbol{E_{otin}}$	other built-in noise sources.
12.	$\boldsymbol{E_{otex}}$	other external noises sources like external resistors

The total noise power is computed as sum of all noise contributions. Example:

$$E_{\text{int}}^2 = E_q^2 + E_{nonlin}^2 + E_{switch}^2 + E_{thermal}^2 + E_{pink}^2 + E_{otin}^2, \qquad E_{\text{int}} = E_{\text{int},rms} = \sqrt{E_{\text{int}}^2}$$

$$E_{ext}^2 = E_{alias}^2 + E_{T\&H}^2 + E_{clkj}^2 + E_{clkft}^2 + E_{current}^2 + E_{otex}^2, \qquad E_{ext} = E_{ext,rms} = \sqrt{E_{ext}^2}$$

$$E_{tot}^2 = E_{\text{int}}^2 + E_{ext}^2, \qquad E_{tot} = E_{tot,rms} = \sqrt{E_{\text{int}}^2 + E_{ext}^2}$$

In the following, we will use a sinusoidal test signal with the maximum possible amplitude, so that *SNR* (Signal-to-Noise Ratio) and *SINAD* (Signal to Noise & Distortion ratio) are the same.

Typically, we have a system accuracy goal given by the specifications:

$$SNR_{tot} = \frac{\text{Sin}\,usoidalSignalPower}{TotalNoisePower} = \frac{U_{S,rms}^2}{E_{tot}^2} = \frac{U_{S,rms}^2}{E_{\text{int}}^2 + E_{ext}^2} = \frac{R^2/8}{E_{\text{int}}^2 + E_{ext}^2},$$

with *R* being the peak-to-peak voltage range. Power data in dB cannot be added, so we have to compute absolute power data. There are several possibilities to translate SNR_{dB} to *SNR*, which is a power-ratio factor:

$$SNR = 10^{\frac{SNR_{dB}}{10dB}} = 2^{\frac{SNR_{dB}}{3.01dB}} = 2^{2(ENoB+1.76)}$$

With sinusoidal signal power $R^2/8$, we calculate the available total noise power budget as

$$\boxed{E_{tot}^2 = \frac{SignalPower}{SNR} = \frac{R^2/8}{10^{\frac{SNR_{dB,tot}}{10dB}}}}.$$

With a vendor-given, device dependent $SNR_{dB,int}$ (or $SINAD_{dB,int}$) we get

$$\boxed{E_{\text{int}}^2 = \frac{R^2/8}{10^{\frac{SNR_{dB,\text{int}}}{10dB}}}}.$$

The remaining noise power budged "external" of our conversion device is

$$\boxed{E_{ext}^2 = E_{tot}^2 - E_{\text{int}}^2} = \frac{R^2}{8}\left(\frac{1}{SNR_{tot}} - \frac{1}{SNR_{\text{int}}}\right) = \frac{R^2}{8}\left(\frac{1}{10^{\frac{SNR_{dB,tot}}{10dB}}} - \frac{1}{10^{\frac{SNR_{dB,\text{int}}}{10dB}}}\right).$$

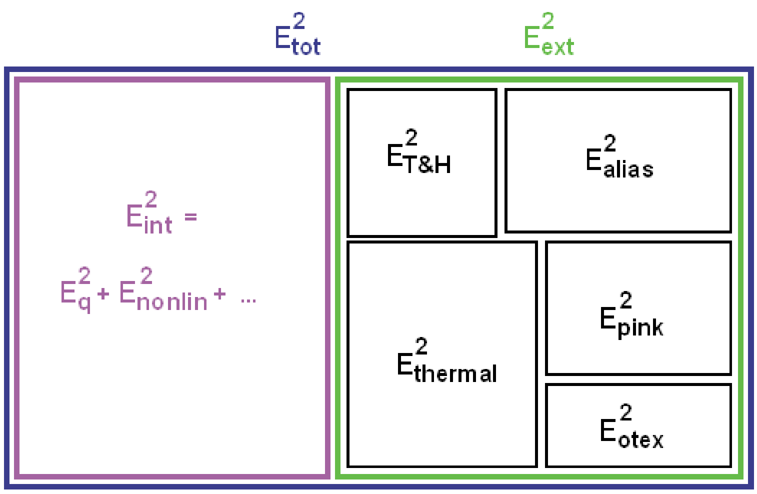


Fig: 5.3
Total noise-power budget E_{tot}^2, its share from the inside the conversion device, E_{int}^2, and the remaining, device-"external" noise-power budget, E_{ext}^2.

If this noise-power budget is assumed to be equi-distributed over *K* noise sources, we get (for example with $xxx \in \{alias, clkj, T\&H, otex\}$)

$$\boxed{E_{xxx}^2 = \frac{E_{ext}^2}{K} \qquad \Leftrightarrow \qquad E_{xxx,rms} = \sqrt{E_{xxx}^2} = \frac{E_{ext,rms}}{\sqrt{K}}}\,.$$

Example:
Given is a 2.7V technology. Required total accuracy is $SNR_{tot,dB}$ = 80 dB. Given by the vendor is $SNR_{dB,int}$ = 83 dB. What is the effective noise voltage budget $E_{ext,rms}$ available for the customer and what is the effective noise-voltage budget $E_{xxx,rms}$ for any of the 4 external noise sources? (The noise budget is to be distributed over the 4 noise sources with same power.)

$$SNR_{tot} = 10^{SNR_{tot,dB}/10dB} = 10^{80dB/10dB} = 10^8$$

$$U_{S,rms}^2 = R^2 / 8 = (2.7\text{V})^2/8 = 0.911\ \text{V}^2$$

$$E_{ext}^2 = U_{S,rms}^2\left(\frac{1}{SNR_{tot}} - \frac{1}{SNR_{\text{int}}}\right) = \frac{R^2}{8}\left(\frac{1}{10^8} - \frac{1}{2\cdot 10^8}\right) = \frac{(2.7V)^2}{8}\frac{1}{2\cdot 10^8} = 4.556\cdot 10^{-9}V^2$$

$$E_{ext,rms} = \sqrt{E_{ext}^2} = \sqrt{4.556\cdot 10^{-9}V^2} = 67.5\mu\text{V} \quad \rightarrow \quad E_{xxx,rms} = E_{ext,rms} / 2 = 33.75\mu\text{V}$$

In the following subsections we will compute the noise power of the different noises sources mentioned above.

Exercise:
Given is a 3.3V technology. Required total accuracy is $SNR_{tot,dB}$=*90dB*. Given by the vendor is $SNR_{int,dB}$=95dB. What is the total effective noise-voltage budget $E_{ext,rms}$ available for the customer and what is the effective noise-voltage budget $E_{xxx,rms}$ for any of the 5 external noise sources? (The noise budget is to be distributed over the 5 noise sources with same power.)

SNR_{tot} =

. . . .

$U_{S,rms}^2$ =

. . . .

E_{ext}^2 =

. . . .

$E_{ext,rms}$ =

. . . .

$E_{xxx,rms}$ =

. . . .

Solution to the exercise above:

$SNR_{tot} = 10^{SNR_{tot,dB}/10dB} = 10^{90dB/10dB} = 10^9$, $U_{S,rms}^2 = R^2 / 8 = (3.3\text{V})^2/8 = 1.361\ \text{V}^2$,

$$E_{ext}^2 = U_{S,rms}^2\left(\frac{1}{SNR_{tot}} - \frac{1}{SNR_{\text{int}}}\right) = \frac{(3.3V)^2}{8}\left(\frac{1}{10^9} - \frac{1}{10^{9.5}}\right) = 9.308\cdot 10^{-10} V^2$$

$E_{ext,rms} = \sqrt{E_{ext}^2} = \sqrt{6.806\cdot 10^{-10} V^2} = 30.51\mu\text{V} \rightarrow E_{xxx,rms} = E_{ext,rms} / \sqrt{5} = 13.64\mu\text{V}$

5.4 Computing Noise Power of the Different Noise Sources

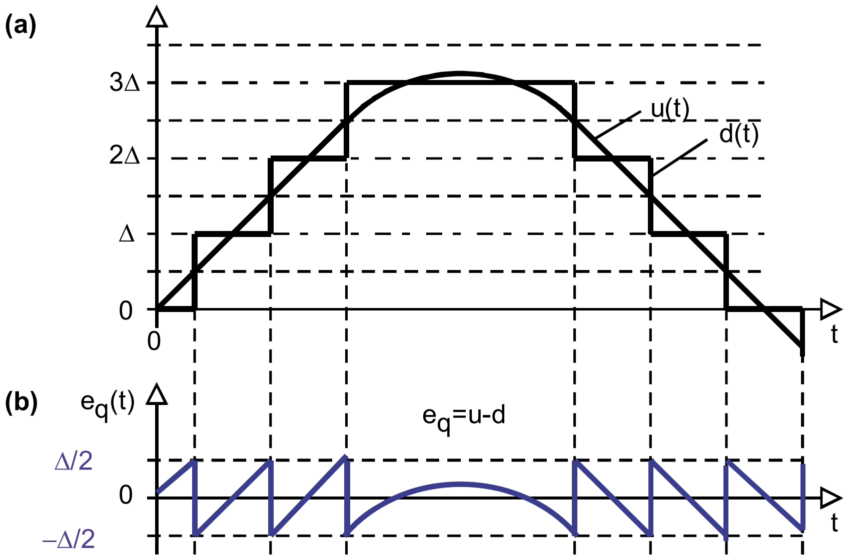
5.4.1 Quantization Noise Power of DAC Output Waveforms

5.4.1.1 Best-Case SNR for Multi-Bit Quantization

Fig. 5.4.1.1:

(a) Multi-bit quantization of a signal $A \cdot sin(\omega t)$ with amplitude $A>\Delta$.

(b) Quantization error $e_q(t)$ has a mostly triangular shape.



Quantization noise is a quantity that mainly depends on the smallest possible step, termed Δ, of an A/D or D/A converter, and it corresponds to numerical round-off noise.

For a sufficiently busy signal with signal range $R_s >> \Delta$, quantization error $e(t)$ has a triangular shape over time axis with range $R_q = \Delta$, as shown in Fig. 5.4.1.1(b). Consequently,

$$E_{q,tria}^2 = \frac{\Delta^2}{12} \qquad \Leftrightarrow \qquad E_{q,tria,rms} = \frac{\Delta}{\sqrt{12}}$$

For a NoB binary input bits DAC with $NoB \geq 10$ we use the approximation

$$\Delta = \frac{R}{2^{NoB}-1} \cong \frac{R}{2^{NoB}}$$

Quantizing $s(t) = (R/2)\cdot sin(\omega t)$, the best obtainable SNR respecting quantization noise only is

$$SNR_q = \frac{S_{rms}^2}{E_{q,tria}^2} = \frac{R^2/8}{\Delta^2/12} = \frac{R^2/8}{R^2/(2^{2NoB}\cdot 12)} = 2^{2NoB}\frac{3}{2}$$

$$SNR_{q,dB} = 10dB\cdot \lg\left(SNR_q\right) \cong NoB\cdot 6.02dB + 1.76dB$$

The factor 3/2 in SNR_q corresponding to 1.76dB in $SNR_{q,dB}$ stems from the fact that reference signal $s(t)=(R/2)\cdot sin(t)$ is sinusoidal and the quantization noise $e_q(t)$ is triangular.

5.4.1.2 Best-Case SNR for Single-Bit Quantization (*NoB*=1)

Pulse-width modulation (PWM) and ΔΣ modulation are frequently used with single-bit quantization. When the transferred signal range R is small compared to Δ, i.e. $R \ll \Delta$, quantization noise can be assumed to be rectangular as illustrated in Fig. 5.4.1.2-1. Then effective power of quantization noise is

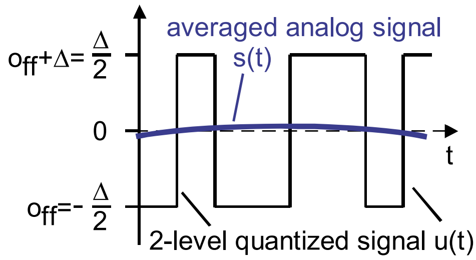


Fig. 5.4.1.2-1: Signal range R << Δ yields rectangular quantization noise

$$E_{q,rect}^2 = \frac{\Delta^2}{4} \quad \Leftrightarrow \quad E_{q,rect,rms}^2 = \frac{\Delta}{2}$$

so that

$$SNR_{q,rect} = \frac{S_{rms}^2}{E_{q,rect}^2} = \frac{R^2/8}{\Delta^2/4} = \frac{R^2/8}{R^2/(2^{2NoB}\cdot 4)} = 2^{2NoB}\frac{1}{2}$$

and

$$SNR_{q,dB} = 10dB \cdot \lg\left(SNR_q\right) \cong NoB \cdot 6.02dB - 3.01dB$$

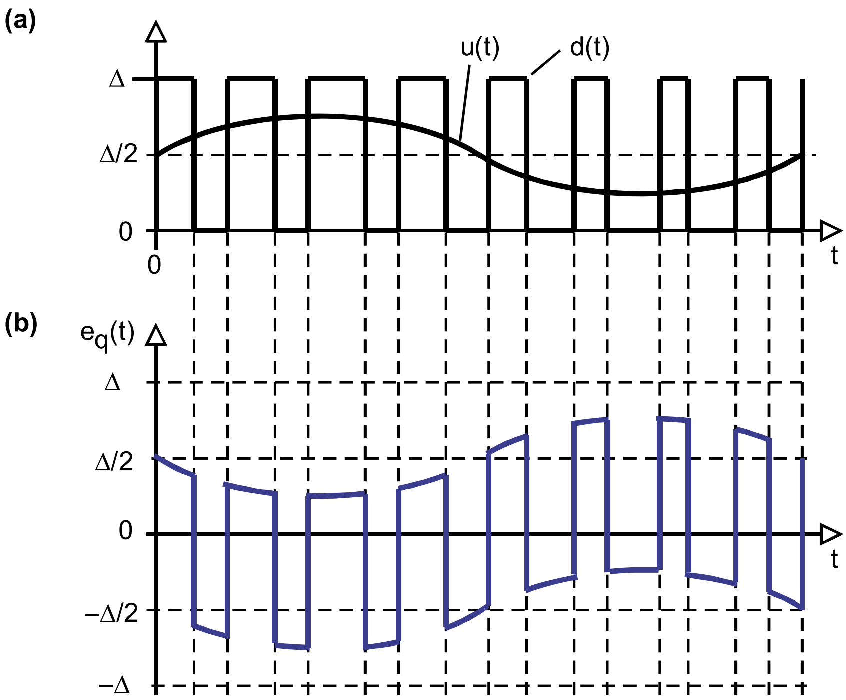
The factor 1/2 in $SNR_{q,rect}$ corresponding to -3.01dB in $SNR_{q,rect,dB}$ stems from the fact, that reference signal $s(t)=(R/2)\cdot sin(t)$ is sinusoidal and the quantization noise $e_q(t)$ is rectangular.

When signal amplitude *A* is similar to rectangular single-bit quantization Δ :

Fig 5.4.1.2-2:
One-Bit quantizer.

(a) signal *u(t)* obtained by averaging *d(t)*.

(b) Quantization noise obtained as difference $e_q(t)=d(t)-u(t)$.



To compute a PWM signal we observe one time interval $T=T_H+T_L$, where T_H is the total high-time during and T_L the total low-time of the signal. We define $u_{rect}(t)$ as

$$u_{rect}(t)=\begin{cases} o_{ff}+\Delta & \text{when} \quad 0\le t-t_i<T_H \\ o_{ff} & \text{when} \quad T_H\le t-t_i<T \end{cases}$$ with offset o_{ff} and $t_i=i\cdot T$, i=0, 1, 2, 3, ...

and

$$s=\frac{1}{T}\int_0^T u_{rect}(t)\cdot dt=\frac{1}{T}\left(\int_0^{T_H}(o_{ff}+\Delta)\cdot dt+\int_{T_H}^{T_H+T_L}x\cdot dt\right)=o_{ff}+\frac{T_H}{T}\Delta .$$

Using

$$D=\frac{T_H}{T_H+T_L}=\frac{T_H}{T} \quad \Leftrightarrow \quad \frac{T_L}{T}=1-D .$$

we get signal s, which is the average of $u_{rect}(t)$, as

$$s=o_{ff}+D\cdot\Delta$$

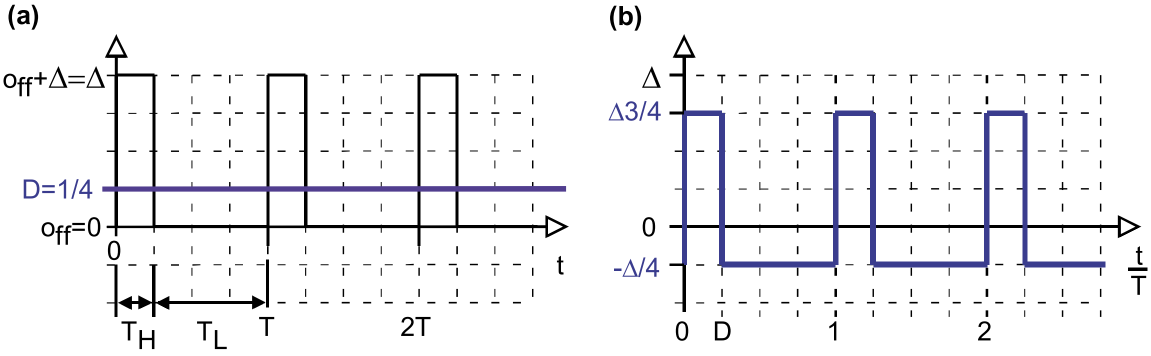


Fig. 5.4.1.2-3: (a) Rectangular signal with average, **(b)** quantization noise = signal – average.

The quantization error is

$$e_{q,rect}(t) = u_{rect}(t) - s = \begin{cases} (\Delta + o_{ff}) - s = & (1-D)\Delta \quad when \quad u_{rect}(t) = o_{ff} + \Delta \\ s - o_{ff} = & D \cdot \Delta \quad when \quad u_{rect}(t) = o_{ff} \end{cases}$$

The total quantization noise power is

$$E_{q,rect}^2 = \frac{1}{T}\int_0^T e_q^2(t) \cdot dt = \frac{\Delta^2}{T}\left(\int_0^{T_H}(1-D)^2 dt + \int_{T_H}^{T_H+T_L} D^2 dt\right) = \Delta^2\left((1-D)^2\frac{T_H}{T} + D^2\frac{T_L}{T}\right)$$

$$= \Delta^2\left((1-D)^2 D + D^2(1-D)\right) = \Delta^2 D(1-D)$$

$$E_{q,rect}^2 = \Delta^2 D(1-D) = \Delta^2(0.25 - s^2)$$

$$E_{q,rect} = \Delta\sqrt{D(1-D)} = \Delta\sqrt{0.25 - s^2}$$

with a maximum at D=0.5.

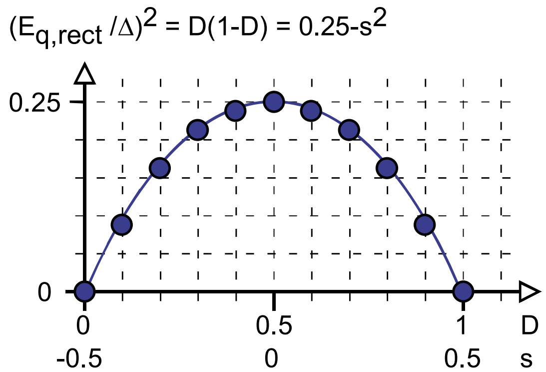


Fig. 5.4.1.2-4: Quantization noise power as function of duty cycle *D*.

For ΔΣ modulators the total High- and Low- times consist of several disjointed bits and the integration interval *T* may not be so clearly to define. However, the result is the same.

5.4.2 Quantization Noise Power of an ADC Samples

5.4.2.1 Multi-Bit Quantization with Sufficiently Busy Input Signal

Let's assume we sample a piece of music that takes 200*s* with a sampling frequency f_S=50KHz. Then we have $N=f_S\cdot 200s=10^7$ samples and the same number of quantization errors $e_i=e_q(t_i)$, where $t_i=i\cdot T=i/f_S$. Their quantization noise power is defined as

$$E_q^2 = \frac{1}{N}\sum_{i=1}^{N} e_i^2 \ .$$

We now arrange the samples e_i according to their size, e_i, and form K groups of width $h=\Delta/K$ containing n_j samples. The we can re-write the sum as

$$E_q^2 \cong \frac{1}{N}\sum_{j=1}^{k} n_j e_j^2 = \sum_{j=1}^{k}\frac{n_j}{N} e_j^2 = \sum_{j=1}^{k} w_j e_j^2 \ .$$

with weight function $w_j=n_j/N$ and $\sum_{j=1}^{K} w_j = 1$. For $h\rightarrow 0$ this sum strives to

$$E_q^2 = \int_{-\Delta/2}^{\Delta/2} w(e_q)\cdot e_q^2 \cdot de_q$$

As the total probability $\int_{-\infty}^{\infty} w(e_q)\cdot de_q = 1$. Assuming ***$e_i$* uniformly distributed** within the interval -Δ/2 ... Δ/2 the shape of $w(e_q)$ is given by

$$w(e_q) = \begin{cases} 1/\Delta & when \quad |e_q| < \Delta/2 \\ 0 & otherwise \end{cases}$$

and the integral evaluates to

$$E_q^2 = \int_{-\infty}^{\infty} w(e_q)\cdot e_q^2 \cdot de_q = \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta}\cdot e_q^2 \cdot de_q = \frac{\Delta^2}{12}$$

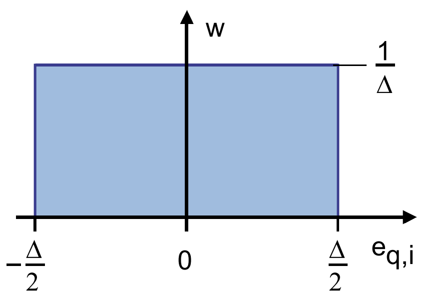


Fig. 5.4.2.1: Probability $w(e_q)$ is uniformly distributed over e_q. I.e. any e_q has the same probability to occur within interval -Δ/2 ... Δ/2.

In summary

$$\boxed{E_q^2 = \frac{\Delta^2}{12}} \quad \Leftrightarrow \quad \boxed{E_{q,rms} = \sqrt{E_q^2} = \frac{\Delta}{\sqrt{12}}} \ .$$

Note that this is exactly the same result as for the triangular, time-continuous output waveform of the DAC. In fact, if we would order all the e_i by size they would form a triangle.

5.4.2.2 Single Bit (*NoB*=1) Toggles at Constant Input Signal

This happens typically if only the least significant bit (LSB) oscillates. The signal is computed by averaging the in put samples $s_i=s(t_i)$. We have a total number of N samples, with n_L of them having the value $s_i=0$ and n_H of them the value $s_j=\Delta$. Their average value (here assumed to be more or less constant with respect to the sampling rate) is computed as averaging value:

$$s(t)=\frac{1}{N}\sum_{i=1}^{N}s_i\,.$$

Using

$$D=\frac{n_H}{N}\quad\Leftrightarrow\quad 1-D=\frac{n_L}{N}$$

delivers the average signal as

$$s(t)=D\cdot\Delta\,.$$

Consequently, we have n_L errors of size $e_{q,i}=s_i-s(t)=-D\Delta$ and n_H errors of size $e_{q,j}=s_j-s(t)=(1-D)\Delta$. The total noise power is then

$$E_q^2=\frac{1}{N}\sum_{i=1}^{N}e_{q,i}^2=\frac{n_L}{N}(D\Delta)^2+\frac{n_H}{N}((1-D)\Delta)^2=(1-D)(D\Delta)^2+D((1-D)\Delta)^2=D(1-D)\Delta^2$$

The final result,

$$E_q^2=D(1-D)\Delta^2\quad\text{with its maximum of }E_q^2=\frac{\Delta^2}{4}\text{ at }D=\frac{1}{2}.$$

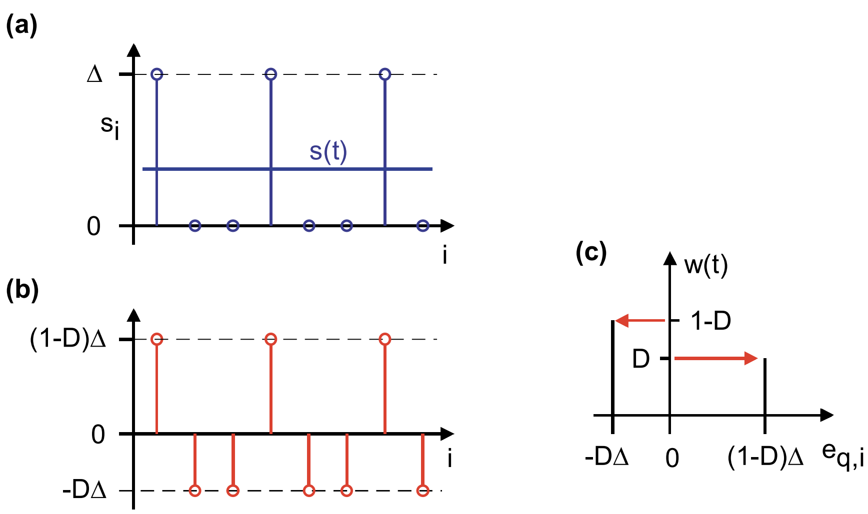
is essentially the same as we had for the 2-level DAC in the time-continuous regime. In fact, we could reorder our $e_{q,i}$ to form the sampling of a pulse-width modulated wave.

Fig. 5.4.2.2:

(a) Samples s_i and their average value $s(t)$.

(b) Quantization errors computed from $e_{q,i}=s_i-s(t)$.

(c) Probability of quantization errors $e_{q,i}$ to occur.



5.4.3 Quantization Noise Power in the Frequency-Domain

5.4.3.1 Using Shape Functions to Model the Frequency Domain View.

We now know from time-domain considerations that the total quantization noise power of a multi-bit quantization of a sufficiently busy signal *s(t)* with amplitude $A >> \Delta$ delivers the quantization-noise power

$$E_q^2 = \frac{\Delta^2}{12}$$

with Δ being the least significant bit. We can use $e_q(t)$ or $e_{q,i}$, $i=1...N$, to compute the Fourier transformed of this functions. As the Fourier transformation must be done for a particular signal, we construct a general approximation that we can transform. Respecting that different frequencies are principally uncorrelated we have to add or integrate them in power:

$$E_q^2(f) = \int_{\xi=0}^{f} E_q^{2\,'}(\xi) \cdot d\xi \qquad \Leftrightarrow \qquad E_q^{2\,'}(f) = \frac{d}{df} E_q^2(f)$$

where the abbreviation E_q^2 stands for

$$E_q^2 = E_q^2(f)\Big|_{f\to\infty} = \frac{\Delta^2}{12}.$$

Considering spectral quantization noise the range f=0...f_s/2, with f_S being the sampling frequency, we can write

$$E_q^{2\,'}(f) = E_q^2 \cdot w(f)$$

with shape function *w(f)* having the property

$$\int_{-\infty}^{+\infty} w(f)df = 1$$

In the time-discrete domain we use a sampling frequency f_S and the relative frequency $F = \dfrac{f}{f_S}$. Consequently, dF/df = $1/f_s$ yields df = f_S dF and the shape integral translates to

$$\int_{f_1}^{f_2} w(f) \cdot df = \int_{F_1}^{F_2} W(F) \cdot dF$$

with $F_1=f_1/f_S$, $F_2=f_2/f_S$ and $W(F) = f_S \cdot w(f / f_S)$.

5.4.3.2 Quantization Noise at Nyquist Sampling : Bandwidth $f_B = ½ f_S$.

Given is sampling frequency f_S and consequently
Absolute bandwidth $f_B = f_S/2$,
Relative bandwidth $F_B = f_B/f_S = ½$.

If the signal *s(t)* is *sufficiently busy* with respect to f_S and the quantization process as the same probability to hit any value in the range –Δ/2...Δ/2, then the shape function for a quantization error is "white", i.e.

Table 5.4.3.2: Shape functions with unit area at Nyquist sampling: $f_B = ½ f_S$

Quantity	over real frequency axis f	over relative frequency axis F
shape function	$w(f) = \begin{cases} \frac{2}{f_S} = \frac{1}{f_B} & \text{if} \quad 0 \le f \le \frac{f_S}{2} \\ 0 & \text{otherwise} \end{cases}$	$W(F) = \begin{cases} 2 = \frac{1}{F_B} & \text{if} \quad 0 \le F \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$
spectral quantization noise power	$E_q^{2'}(f) = E_q^2 \cdot w(f)$	$E_q^{2'}(F) = E_q^2 \cdot W(F)$

Integrating the total noise power in the baseband 0…f_B or over 0…F_B obtains the total noise power of the sampler. As this power cannot depend of the kind of integration the result over frequency must be the same as obtained n time-domain, namely E_q^2.

f-axis: $$\int_0^\infty E_q^{2'}(f) \cdot df = \int_{-\infty}^\infty E_q^2 \cdot w(f) \cdot df = E_q^2 \int_0^{f_B} \frac{1}{f_B} \cdot df = E_q^2 \frac{1}{f_B} f_B = E_q^2 \ ,$$

F-axis: $$\int_0^\infty E_q^{2'}(F) \cdot dF = \int_{-\infty}^\infty E_q^2 \cdot W(F) \cdot dF = E_q^2 \int_0^{F_B} \frac{1}{F_B} dF = E_q^2 \cdot \frac{1}{F_B} F_B = E_q^2 \ .$$

5.4.3.3 Quantization Noise Reduction by Simple Oversampling

Sampling frequency f_S is now increased to an oversampling ratio of

$$OSR = \frac{f_S}{2f_B} = \frac{1}{2F_B} \ .$$

Consequently,

$$f_B = \frac{f_s}{2OSR} \qquad \Leftrightarrow \qquad F_B = \frac{1}{2OSR}$$

with $OSR > 1$. Note that Nyquist sampling corresponds to $OSR = 1$. Consequently we have

Table 5.4.3.3: Shape functions with unit area at over-sampling: $f_B = ½ f_S/OSR$

over real frequency f	over relative frequency F
$w(f) = \begin{cases} \frac{2}{f_S} = \frac{1}{OSR \cdot f_B} & if \quad 0 \le f \le \frac{f_S}{2} \\ 0 & otherwise \end{cases}$	$W(F) = \begin{cases} 2 = \frac{1}{OSR \cdot F_B} & if \quad 0 \le F \le \frac{1}{2} \\ 0 & otherwise \end{cases}$
$E_q^{2'}(f) = E_q^2 \cdot w(f)$	$E_q^{2'}(F) = E_q^2 \cdot W(F)$

Integrating the total noise power in the baseband $0 \ldots f_B$ or $0 \ldots F_B$ delivers the total noise power in the baseband.

f-axis: $$\int_0^\infty E_q^{2'}(f) \cdot df = E_q^2 \int_{-\infty}^{\infty} w(f) \cdot df = E_q^2 \int_0^{f_B} \frac{1}{OSR \cdot f_B} \cdot df = E_q^2 \frac{1}{OSR \cdot f_B} f_B = \frac{E_q^2}{OSR}$$

F-axis: $$\int_0^\infty E_q^{2'}(F) \cdot dF = E_q^2 \int_{-\infty}^{\infty} W(F) \cdot dF = E_q^2 \int_0^{F_B} \frac{1}{OSR \cdot F_B} dF = E_q^2 \cdot \frac{1}{OSR \cdot F_B} F_B = \frac{E_q^2}{OSR}$$

Consequently, by oversampling with ratio OSR
the noise power within the baseband is reduced with $1/OSR$,
the noise amplitude within the baseband is reduced with $1/\sqrt{OSR}$.

This noise reduction assumes an ideal lowpass, i.e. $|H_{LP}(f)|=\{1$ when $f \le f_B$, 0 otherwise$\}$. With a non-ideal lowpass we obtain

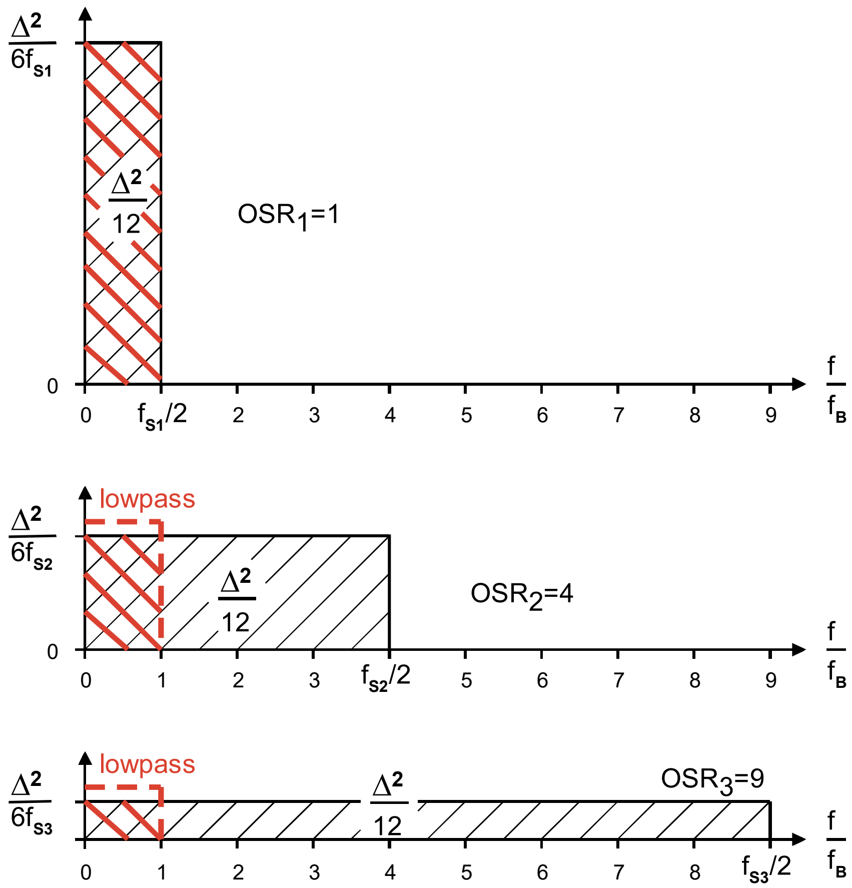
$$\int_{-\infty}^{\infty} E_q^{2'}(f) H_{LP}^2(f) \cdot df \quad \text{over } f \quad \text{or} \quad \int_{-\infty}^{\infty} E_q^{2'}(F) H_{LP}^2(F) \cdot dF \quad \text{over } F.$$

Fig. 5.4.3.3:
White Quantization Noise (0th Order Noise Shaping)

(a) no oversampling

(b) Using an OSR=4 and an ideal lowpass at f_B.

(c) Using an OSR=9 and an ideal lowpass at f_B.



Exercises:
The noise power in your signal has to be reduced by simple oversampling an subsequent filtering with an ideal lowpass with cut-off frequency f_B. Compute the required oversampling ratios:

Reduction of noise ***power*** in the baseband by a factor 10: increase *OSR* by . . . .

Reduction of noise ***amplitude*** by a factor 10: increase *OSR* by . . . .

Improvement of SNR by ***one bit***: increase *OSR* by . . . .

Improvement of SNR by ***NoB bits***: increase *OSR* by . . . .

Improvement of SNR by ***X dB***: increase *OSR* by . . . .

Solutions:
Reduction of noise ***power*** in the baseband by a factor 10: increase *OSR* by **factor 10**
Reduction of noise ***amplitude*** by a factor 10: increase *OSR* by **factor 10^2=100**
Improvement of *SNR* by ***one bit***: increase *OSR* by **$2^2 = 4$ (1 bit is amplitude noise reduction by factor 2)**
Improvement of *SNR* by ***NoB bits***: increase *OSR* by **$2^{2\cdot NOB} = 4^{NOB}$ (*NoB* bits is 2^{NoB} amplitude noise reduction)**
Improvement of *SNR* by ***X dB***: increase *OSR* by **1st way: factor $10^{X/10dB}$ according to definition of dB**
2nd way: factor $4^{X/6.02dB}$ replacing above *NoB* by *NoB*=X/6.02.

5.4.3.4 Quantization Noise Reduction by Noise Shaping and Filtering

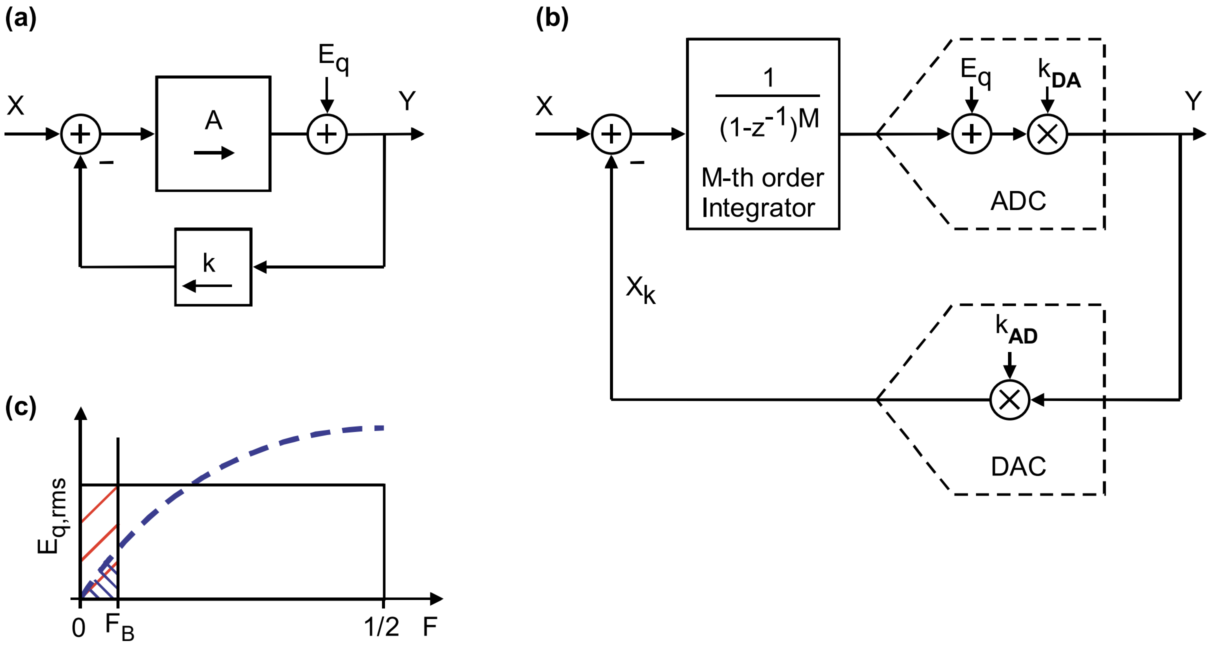
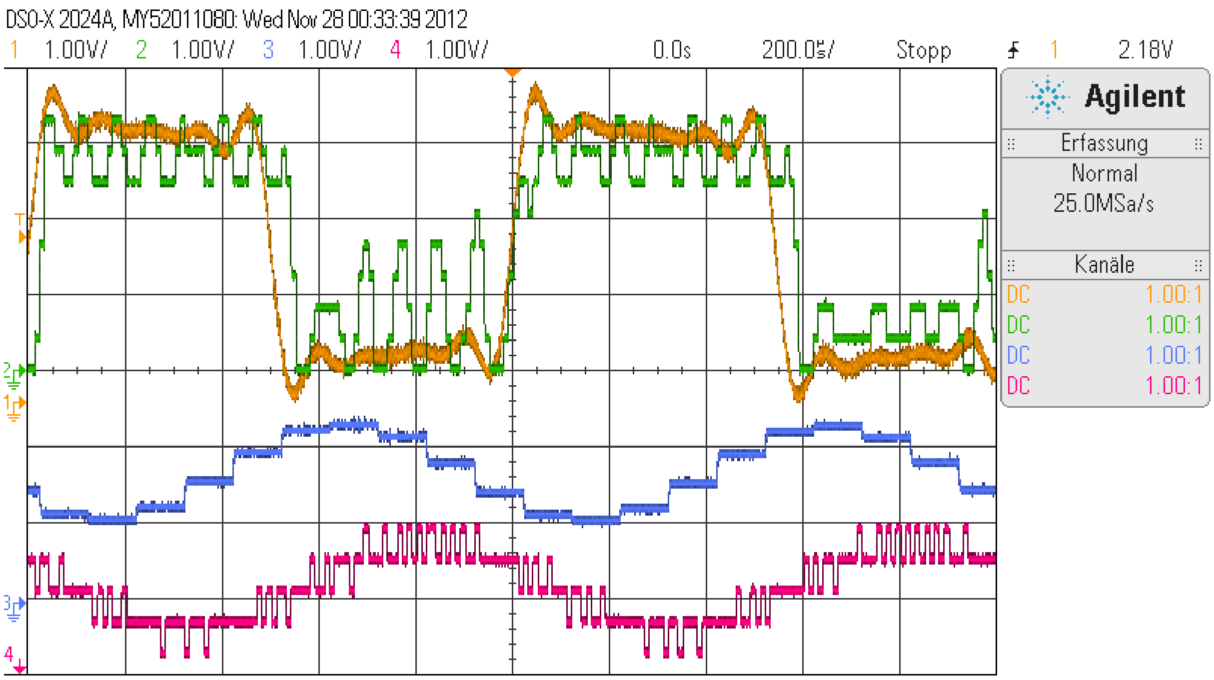


Fig. 5.4.3.4: (a) Feedback loop, **(b)** ΔΣ Modulator, **(c)** shaped noise power $E_{q,rms}(F)$

A ΔΣ modulator consists of an integrator and a quantizer in the forward network and a feedback network which is constant over frequency as shown in the figure above. As the quantizer works time-discrete we use the time discrete integrator model, $1/(1-z^{-1})$.



Screen shot from oscilloscope of 2nd order ΔΣ modulator. Yellow: input signal to ΔΣ ADC. Green: its 9-level modulator's output. Blue: lowpass filtered (=demodulated) green curve. Red: blue curve ΔΣ modulated with 9-level quantizer.

The screen shot above shows:

1. **Yellow**, *X* in the ΔΣ schematics above: Approximately rectangular analog input voltage to a 2nd order modulator (voltage *CP_in_P* of DA2 board of course PRED).
2. **Green**, *Y* in the ΔΣ schematics above: Modulator's output, i.e. the output of the 9-level quantizer 'jumping' fast around the yellow input signal. (To make this digital output visible as analog waveform it was measured at the output X_k of the DAC in the feedback branch as voltage *DAC3out* of DA2 board.)
3. **Blue**, (not in the ΔΣ schematics above): This is the demodulated signal (green) Y. Demodulation is nothing else than lowpass filtering. The blue curve is the output of the ΔΣ ADC. (This originally digital signal and was made visible with a 256-level R2R DAC as voltage *DAC1out* of DA2 board.)
4. **Red**, (not in the ΔΣ schematics above): This is the re-modulated blue ADC output. This was done by a 1st order digital-to-digital ΔΣ modulator with a 9-level quantizer. (This originally digital signal and was made visible here a 9-level DAC as voltage *DAC2out* of DA2 board.)

Illustrating the Power of Oversampling and ΔΣ Modulation

Table 5.4.3.4 below illustrates the power of ΔΣ modulation computing the quantization noise power reduction in the baseband 0...f_B. Oversampling ratio is $OSR=f_s/2{\cdot}f_B$ with sampling frequency f_S. It is assumed that we have quantization noise only and ideal lowpass filters to remove frequencies > f_B.

Example 1: We want to lower quantization noise power in the baseband by 60dB, corresponding to a factor K=1000 in rms voltage or some 10 bits. Obtaining that by simple oversampling requires to increase sampling frequency f_S by a factor $K^2=10^6$. An ideal 1st order ΔΣ modulator could obtain the same SNR improvement with increasing f_S by a factor 126 and an ideal 2nd order modulator could obtain that with an OSR of 27.

Example 2: We have music in the baseband 0...25KHz sampled with f_S=50KHz. Noise power reduction of 60dB in the baseband obtained by plain oversampling requires to increase f_S by a factor $K^2=10^6$ to f_{s0}=50GHz. An ideal 1st order ΔΣ modulator could obtain the same SNR improvement with increasing f_S by a factor 126 to f_{s1}=6.3MHz. An ideal 2nd order modulator could obtain that with an OSR of 27 and consequently f_{s2}=1.35MHz.

Table 5.4.3.4: Theoretically obtainable *SNR* improvements. Taken from [Leme, PhD...]

SNR_{dB}	**20 dB**	**40 dB**	**60 dB**	**80 dB**	**100 dB**
OSR(*order* =2)	5	11	27	67	168
OSR(*order* =1)	6	28	126	578	2657
OSR(*order*=0)	100	10322	1,05E6	106E6	1,08E10

Modulator Overloading

The output of the 1st order ΔΣ modulator shown in the lower (red) curve of the screen shot above has no jump over 2 Δ's, although the 9-level quantizer would make such jumps possible. This output could be realized with a 2-level quantizer with no further problems.

Observe the output of the 2nd order A/D modulator, i.e. the green curve in the screen shot above. While the output signal is more or less constant we find jumps over 2 Δ's. This is because a 2nd order modulator required 2Δ-jumps. A 2-level quantizer offering Δ-jumps only is said to be overloaded. But it is still stable.

Increasing the order of a ΔΣ modulator pushes more noise to higher frequencies. This can be observed by jumps over several Δ's. If the modulator needs to jump over more deltas than the quantizer can realize we call this overloading. A 2nd order modulator is still stable with a 1-bit (=2-level) quantizer and delivers relatively good results. Higher order modulators become unstable and loose accuracy in case of overloading.

ΔΣ Noise Power Reduction in Baseband 0…f_B :

Key message: The quantization noise in the baseband 0...f_B with $OSR=f_S/2f_B$, and K_{order} being a constant, we get

$$\boxed{E_{q,B}^2 = \frac{K_{order}^2}{OSR^{2order+1}} \quad <=> \quad E_{q,B,rms} = \frac{K_{order}}{OSR^{order+1/2}}}$$

The noise power within baseband $f=0...f_B$ is reduced proportional $1/OSR^{2\cdot order+1}$.

Predication 1, using M = ΔΣ modulator's order (see proof 1 below:
A ΔΣ modulator with M-th order integrator has a constant signal transfer function (namely $STF=1/k_{AD}$) and a noise power spectrum shaped according to $\mathbf{2C_M \cdot \sin^{2\cdot order}(\pi F)}$ over relative frequency $F=f/f_S$, with C_{order} being a constant. Figure 5.4.3.4(c) illustrates first order shaping of effective error amplitudes $E_{q,rms}$.

Predication 2 (see proof 2 below):
The noise shaping can be quantified as

$$E_q^{2'}(F) = E_q^2 \cdot 2C_M \sin^{2M}(\pi F) \quad \text{with} \quad C_M = \frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2M}{2M-1} = 2\cdot\prod_{k=1}^{M}\frac{2k}{2k-1}$$

Predication 3 (see proof 3 below):
The total quantization noise power E_q^2 and amplitude $E_{q,rms} = \sqrt{E_q^2}$, that is generated by the quantizer. Its part within the based $f=0\ldots f_B$ (corresponding to $F=0\ldots F_B$) is reduced to:

$$E_{q,B}^2 = \frac{E_q^2 C_M}{2M+1}\left(\frac{\pi}{2}\right)^{2M}\frac{1}{OSR^{2M+1}} \quad = \quad E_{q,B,rms} = E_{q,rms}\sqrt{\frac{C_M}{2M+1}}\left(\frac{\pi}{2}\right)^{M}\frac{1}{OSR^{M+0.5}}$$

Application to 1st Order ΔΣ Modulator: *M*=1

C_1=2 and hence

$$E_{q,B}^2 = \frac{E_q^2 C_M}{2M+1}\left(\frac{\pi}{2}\right)^{2M} \frac{1}{OSR^{2M+1}} = E_q^2 \frac{2}{3}\left(\frac{\pi}{2}\right)^2 \frac{1}{OSR^3}$$

and hence

$$E_{q,B,rms} = \sqrt{E_{q,B}^2} = E_{q,rms}\sqrt{\frac{2}{3}}\frac{\pi}{2}\frac{1}{OSR^{1.5}}$$

Note

(1) This is the quantization noise power in the baseband f=0...f_B. A non-ideal lowpass will allow more noise power to pass.

(2) The output of a 1st order ΔΣ modulator performs jumps over one Δ. Therefore, a single-bit output is well.

Application to 2nd Order ΔΣ Modulator: M=2

C_2=8/3 and hence

$$E_{q,B}^2 = \frac{E_q^2 C_M}{2M+1}\left(\frac{\pi}{2}\right)^{2M} \frac{1}{OSR^{2M+1}} = E_q^2 \frac{8}{15}\left(\frac{\pi}{2}\right)^4 \frac{1}{OSR^5}$$

and hence

$$E_{q,B,rms} = \sqrt{E_{q,B}^2} = E_{q,rms}\sqrt{\frac{8}{15}}\left(\frac{\pi}{2}\right)^2 \frac{1}{OSR^{2.5}}$$

Note

(1) This is the quantization noise power in the baseband f=0...f_B. A non-ideal lowpass will allow more noise power to pass.

(2) The output of a 2nd order ΔΣ modulator performs also jumps over 2Δ. Therefore, a single-bit output is called <u>overloaded</u> but still works stable and surprisingly well.

Application to Higher Order ΔΣ Modulator: M≥3,

Higher order modulators are difficult to construct and generate considerable high-frequency noises power with jumps over several Δ's. Particularly when the output is overloaded severe stability problems must be solved [Norsworthy,Schreier,Temes: "ΔΣ Data Converters"].

Example:

We want 1 bit more accuracy for our oversampling ADC. It is demodulated by an ideal lowpass with cut-off frequency f_B. Instead of improving the quantizer we increase the *OSR*. By how much must the *OSR* be increased for M^{th} order (M=0,1,2) ΔΣ modulation? (We assume that there are no other noise sources than quantization noise available.)

1 bit is a factor 2 in voltage and consequently a factor 4 in power.

0th order modulation (M=0, is no modulation and no noise shaping, could be PWM):

$\frac{1}{4} = \frac{1}{OSR}$ => increase *OSR* by factor 4.

1st order ΔΣ modulator (M=1):

$\frac{1}{4} = \frac{1}{OSR^3}$ => increase *OSR* by factor $\sqrt[3]{4} \cong 1.59$.

2nd order ΔΣ modulator (M=2):

$\frac{1}{4} = \frac{1}{OSR^5}$ => increase *OSR* by factor $\sqrt[5]{4} \cong 1.32$.

Exercise 1:

We want 20dB more accuracy for our oversampling ADC. It is demodulated by an ideal lowpass with cut-off frequency f_B. Instead of improving the quantizer we increase the *OSR*. By how much must the *OSR* be increased for M^{th} order (M=0,1,2) ΔΣ modulation? (Assume that there are no other noise sources than quantization noise.)

Exercise 2:

Prove with Matlab that $\int_{F=0}^{1/2} 2C_M \sin^{2M}(\pi F) \cdot dF = 1$, when $C_M = \prod_{k=1}^{M} \frac{2k}{2k-1}$.

Solution to Exercise 1 (M = ΔΣ modulator's order):

We want 20dB more accuracy for our oversampling ADC. It is demodulated by an ideal lowpass with cut-off frequency f_B. Instead of improving the quantizer we increase the *OSR*. By how much must the *OSR* be increased for M^{th} order (M=0,1,2) ΔΣ modulation? (Assume that there are no other noise sources than quantization noise available.)

20dB is – by definition – a factor 100 in power.

0th order modulation (M=0, is no noise shaping):

$$\frac{1}{100} = \frac{1}{OSR}$$ **=> increase *OSR* by factor 100.**

1st order ΔΣ modulator (M=1):

$$\frac{1}{100} = \frac{1}{OSR^3}$$ **=> increase *OSR* by factor** $\sqrt[3]{100} \cong 4.64$.

2nd order ΔΣ modulator (M=2):

$$\frac{1}{100} = \frac{1}{OSR^5}$$ **=> increase *OSR* by factor** $\sqrt[5]{100} \cong 2.51$.

Solution to Exercise 2:

```
% Prove that integral(2CM*sin(pi*F)^2M) == 1
h=1e-5; F=0:h:0.5; Mmax=14;
W=sin(pi*F); W2=W.*W;
W2M=W2;
C(1)=(2/1);                % CM
I(1)=2*C(1)*trapz(W2M)*h; % Integral using trapezoidal rule
plot(F,W2M); hold on, grid on;
for M=2:Mmax;
   C(M)=C(M-1)*(2*M)/(2*M-1); W2M=W2M.*W2;
   I(M)=2*C(M)*trapz(W2M)*h;
   plot(F,W2M);
end; hold off;
C, I % plot CM and I(ntegral)
```

Proof 1: Noise amplitude shaping according to sin^M(πF)
Fig. part (a) above illustrates the typical feedback loop with signal and noise transfer functions

$$STF = \frac{Y}{X} = \frac{A}{1+kA} \xrightarrow{|kA|\to\infty} \frac{1}{k},$$

$$NTF = \frac{Y}{E} = \frac{1}{1+kA} \xrightarrow{|kA|\to\infty} 0.$$

Applying that to the Delta-Sigma modulator in Fig. part (b) delivers

$$STF = \frac{Y}{X} = \frac{A(z)}{1+k_{DA}A(z)} \xrightarrow{|k_{DA}A(z)|\to\infty} \frac{1}{k_{DA}},$$

$$NTF = \frac{Y}{E_q} = \frac{1}{1+k_{DA}A(z)} \xrightarrow{|k_{DA}A(z)|\to\infty} \frac{1}{k_{DA}A(z)} \to 0,$$

with k_{DA} being the amplification of the DAC (e.g. in V/bit). Using the ADC's amplification k_{AD} (e.g. in bit/V) the forward network is given by

$$A(z) = \frac{k_{AD}}{(1+z^{-1})^M}$$

where 1/(1-z^{-1}) models the time-discrete integrator. Consequently the NTF can be modeled as

$$NTF = \xrightarrow{|k_{DA}A(z)|\to\infty} \frac{1}{k_{DA}A(z)} = \frac{(1-z^{-1})^M}{k_{AD}k_{DA}}.$$

Using $k_{ADA} = k_{AD}k_{DA}$ (which is a dimensionless constant) delivers

$$NTF = \xrightarrow{|k_{DA}A(z)|\to\infty} \frac{(1-z^{-1})^M}{k_{ADA}} = \frac{z^{-M/2}}{k_{ADA}}\left(z^{1/2} - z^{-1/2}\right)^M.$$

Using $z = e^{j\omega T} = e^{j2\pi fT} = e^{j2\pi F}$ delivers

$$NTF = \xrightarrow{|k_{DA}A(z)|\to\infty} \frac{z^{-M/2}}{k_{ADA}}\left(z^{1/2} - z^{-1/2}\right)^M = \frac{z^{-M/2}}{k_{ADA}}\left(e^{j\pi F} - z^{-j\pi F}\right)^M = 2j\frac{z^{-M/2}}{k_{ADA}}\sin^M(\pi F).$$

As we are only interested in amplitudes of noise shaping only we use a constant K_M to write

$$|NTF| = \xrightarrow{F\to 0} K_M \cdot \sin^M(\pi F).$$

Proof 2: Noise shaping according to sin^{2M}(πF)

The compute the constant factor K_M in the formula above, we recall to mind that the total quantization noise power E_q^2 is given by time domain considerations. So we write

$$E_{q,\Delta\Sigma}^2(F) = E_q^2 \cdot W_{\Delta\Sigma}(F)$$

with shape function $W_{\Delta\Sigma}(F)=2C_M\cdot\sin^{2M}(\pi F)$, where constant C_M was selected such, that

$$\int_{F=0}^{1/2} W_{\Delta\Sigma}(F)\cdot dF = \int_{F=0}^{1/2} 2C_M \sin^{2M}(\pi F)\cdot dF = 1\,.$$

It can be shown that [Bronstein Semendjajew]

$$C_M = \frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2M}{2M-1} = \prod_{k=1}^{M}\frac{2k}{2k-1}$$

Proof 3: Noise power reduction is according to 1/*OSR*$^{2M+1}$

As ΔΣ modulator is based on oversampling with $OSR=f_S/(2f_B) > 1$, the noise power in the baseband $F=0...F_B$ (corresponding to real frequencies $f=0...f_B$) is given by

$$E_{q,B}^2 = E_q^2\cdot\int_{F=0}^{F_B} W_{\Delta\Sigma}(F)\cdot dF = E_q^2 2C_M\int_{F=0}^{F_B}\sin^{2M}(\pi F)\cdot dF\,.$$

As the integral over $\sin^{2M}$ for any F_B is difficult to evaluate we assume for sufficiently large $OSR=1/2F_B$ the approximation $\sin(x)\approx x$ for $x<<1$. In this case the integral above becomes

$$E_{q,B}^2 = E_q^2 2C_M\int_{F=0}^{F_B}\sin^{2M}(\pi F)\cdot dF \cong E_q^2 2C_M\int_{F=0}^{F_B}(\pi F)^{2M}\cdot dF = \frac{E_q^2 2C_M\pi^{2M}}{2M+1}F_B^{2M+1}$$

The substitution $F_B=1/(2\cdot OSR)$ delivers the desired dependency on the OSR:

$$E_{q,B}^2 = \frac{E_q^2 C_M}{2M+1}\left(\frac{\pi}{2}\right)^{2M}\frac{1}{OSR^{2M+1}} \qquad = \qquad E_{q,B,rms} = E_{q,rms}\sqrt{\frac{C_M}{2M+1}}\left(\frac{\pi}{2}\right)^{M}\frac{1}{OSR^{M+0.5}}$$

5.4.4 E_{nonlin} : Noise Due to Non-Linearity

Non-linearity are errors that do not occur randomly but always for certain input voltages. Their effect is visible in parameters like *THD* (total harmonic distortion, German: Klirrfaktor), *SFDR* (spurious free dynamic range), *INL* and *DNL* (integral and differential non-linearity, respectively).

Non-linearity is a noise source that typically cannot be improved by the customer of an A/D or D/A converter and should therefore be included in the data-sheets best-case *SNR* of the ADC or DAC.

Best-case models:

DAC: $U_{DAC,out} = \Delta_{DA} \cdot N_{DAC,in}$ ADC: $N_{ADC,out} = round(U_{ADC,in} / \Delta_{AD})$

Best-case models with offset voltages:

DAC: $U_{DAC,out} = \Delta_{DA} \cdot N_{DAC,in} + U_{off,DA}$ ADC: $N_{ADC,out} = round(U_{ADC,in} - U_{off,AD}) / \Delta_{AD})$

Examples for including non-linearity:

ADC: $N_{ADC,out} = round\left(\alpha_0 + \alpha_1 U_{ADC,in} + \alpha_2 \cdot U_{ADC,in}^2 + \alpha_3 \cdot U_{ADC,in}^3 + ...\right)$

DAC: $U_{DAC,out} = \Delta_0 + \Delta_1 \cdot N_{DAC,in} + \Delta_2 N_{DAC,in}^2 + \Delta_3 N_{DAC,in}^3 + ...$,

Furthermore we can introduce missing codes, e.g. by

if($N \geq 316$ ***and*** $N \leq 512$) ***then*** N=512 ***end if***;

Non-linearities apper as harmonic distortion. The noise generated by harmonics is measured as total harmonic distortion (THD) as defined in chapter 2 as

$$THD = \frac{P_h}{P_1} = \frac{\sum_{k=2}^{N} \left|X(f_k)\right|^2}{\left|X(f_1)\right|^2} \quad \Leftrightarrow \quad E_{THD}^2 = \sum_{k=2}^{N} \left|X(f_k)\right|^2 = THD \cdot \left|X(f_k)\right|^2 = THD \cdot \frac{R^2}{8}$$

$$THD_{dB} = 10dB \cdot \log_{10}\left(THD\right)$$

whereas X is an amplitude like voltage or current and E_{THD} is the effective (rms) voltage of the harmonics, so we could also write $E_{THD,rms}$.

PS: In audio applications we frequently find the definition

$$THD_{\%audio} = 100\% \cdot \sqrt{THD}\ .$$

5.4.5 E_{otin} : Other Internal Noise Sources

There are several other internal noises sources as Johnson noise of resistors or $1/f$ (=pink) noise of FETs etc. However, it is up to the manufacturer to measure these noise sources and respect them in the best-case *SNR* of the ADC or DAC in the data sheet.

5.4.6 E_{alias} : Alias Noise

External Noise. This is a noise that can typically be attenuated by external lowpass filters.

In the **a**nalog domain we use indices ***A*** and *B*, for attenuation and bandwidth frequencies, in the **d**igital domain we use indices *C* and ***D*** for cutoff and damping frequencies, respectively.

5.4.6.1 Computing Alias Frequencies

Assuming a real sampling frequency f_S we define the relative frequency $F=f/f_S$. Due to the theorems of Nyquist and Shannon the maximum frequency that can be represented is

$$f = 0 \ldots \tfrac{1}{2} f_S \qquad \Leftrightarrow \qquad F = 0 \ldots \tfrac{1}{2}$$

If we sample frequencies higher than $\tfrac{1}{2} f_S$ does not obtain lowpass filtering but causes aliasing, i.e. the sampled frequency is observed at

$$f_{alias} = | f - f_S \,\mathrm{round}(f/f_S) | \qquad \Leftrightarrow \qquad F_{alias} = | F - \mathrm{round}(F) |.$$

This might yield negative frequencies corresponding to a phase shift. The amplitudes of original and alias signals are the same.

Exercise:

A typical sampling frequency for telephones is 8 KHz. Note at what alias frequencies the following frequencies of the original voice will appear:

Original frequency / KHz	→	Alias frequency / KHz
0.5	→	
2	→	
3.5	→	
4.5	→	
9	→	
13	→	
22	→	

Solution:

A typical telephone sampling frequency is 8KHz. Note on what alias frequencies the following frequencies of the original voice will appear:

Original frequency / KHz	→	Alias frequency / KHz
0.5	→	|0.5-0| = 0.5
2	→	|2-0| = 2
3.5	→	|3.5-0| = 3.5
4.5	→	|4.5-8| = 3.5
9	→	|9-8| = 1
13	→	|13-2·8| = 3
22	→	|22-3·8| = 2

5.4.6.2 Required Anti-Alias Attenuation

It is clear that these alias frequencies are perceived as noise and have to be removed ***before*** sampling. When we want to suppress aliasing noise by X or X_{dB} dB, then the anti-aliasing lowpass has to this attenuation at the $f_s/2$.

Next we check for our aliasing-noise-power budget $E^2_{alias,\max}$ and compare it to the maximum possible power of a sinusoidal signal swinging in the range R, which has the power $R^2 / 8$. If this signal is subject to aliasing its power has to be attenuated to

$$\boxed{X = \frac{R^2/8}{E^2_{alias,\max}}} \quad \Leftrightarrow \quad \boxed{X_{dB} = 10dB \cdot \lg\left(\frac{R^2/8}{E^2_{alias,\max}}\right) = 20dB \cdot \lg\left(\frac{R/\sqrt{8}}{E_{alias,\max}}\right)}$$

5.4.6.3 Required Order of Anti-Aliasing Filters

Bandwidth f_B denotes the filter's pass-band. Stop-band attenuation X is guaranteed for $f \geq f_A$ and $X_{dB}=20\cdot\log_{10}(X)$.

Assume equal amplitudes at the filter's input. At the filter's output U_B is the pass-band amplitude and U_A the attenuated stop-band amplitude. We should get $U_A \leq X \cdot U_B$.

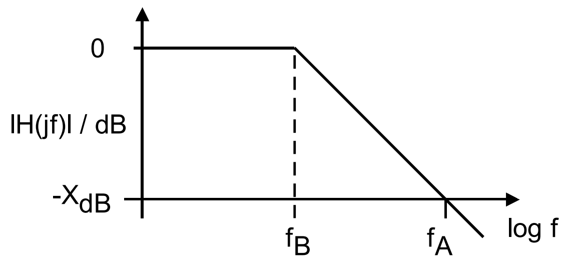


Fig. 5.4.6.3: lowpass asymptotes.

Abbreviating *log₁₀* with *lg* we can write the required filter order

$$N \geq \left|\frac{\lg(U_A) - \lg(U_B)}{\lg(f_A) - \lg(f_B)}\right| = \left|\frac{\lg(U_A/U_B)}{\lg(f_A/f_B)}\right| = \frac{|X_{dB}|}{20dB \cdot \lg\frac{f_A}{f_B}}$$

With function *ceil* for rounding up the minimum filter order is

$$\boxed{N = ceil\left(\frac{X_{dB}}{20dB \cdot \lg\frac{f_A}{f_B}}\right)}$$

Example:

$X_{dB} = 60$dB, f_B=2KHz, $f_A = 4$ KHz. Compute the required filter order N:

***N* ≥ ceil(60dB/(20dB·lg(4Khz/2KHz))) = ceil(9,965) = 10.**

Exercise 1:

$X_{dB} = 60$dB, f_B=3KHz, $f_A = 4$ KHz. Compute the required filter order N:

. . . .

Exercise 2:

X_{dB} = 60dB, f_B=3.7KHz, Nyquist sampling, f_S = 8 KHz. Compute the required filter order N:

. . . .

Exercise 3: Given is a 8-bit ADC. Aliasing noise power must not exceed the quantization noise power of a half LSB. What is the required attenuation of aliasing frequencies?

. . . .

Practical Comment: If anti-aliasing filtering is necessary, a Butterworth filter is appropriate. It has a flat baseband transfer function and –3dB attenuation in the asymptote's kink at f_B, independently from filter order N.

Butterworth lowpass transfer function: $|H_{BW}(jf)| = \dfrac{1}{\sqrt{1+(f/f_B)^{2N}}}$.

Solution to exercise 1: X_{dB} = -60dB, f_B=2KHz, f_A = 8 KHz. Compute the required filter order N.

N ≥ ceil(60dB/(20dB·lg(4Khz/3KHz))) = ceil(24.01) → N = 24.

Solution to exercise 2: X_{dB} = 60dB, f_B=3.7KHz, f_S = 8 KHz. Compute the required filter order N:

N ≥ ceil(60dB/(20dB·lg(½·8Khz/3.7KHz))) = ceil(88.6) → N = 89.

Solution to exercise 3: Given is a 8-bit ADC. Aliasing noise power must not exceed the quantization noise power of a half LSB. What is the required attenuation of aliasing frequencies?
Required attenuation: X_{dB} = -(9 x 6.02 + 1,76)dB = -55.94 dB

5.4.6.4 Matching Analog Anti-Alias and Digital Lowpass Filters

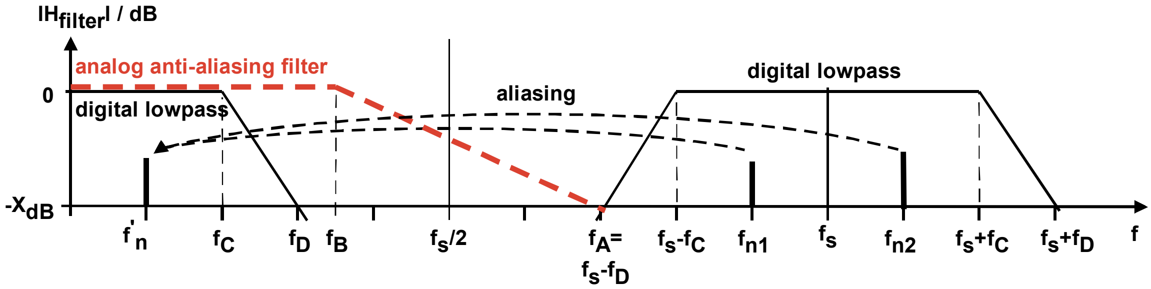


Fig. 5.4.6.4-1: Necessity for an analog anti-aliasing filter: Guarantee sufficient attenuation at f_A=f_S-f_D to suppress aliasing, e.g. from f_n to f'_n.

Today we have a strong tendency to replace analog circuitry by digital circuitry if possible. The figure above illustrates how to relax analog anti-aliasing filters by oversampling and subsequent digital filtering. Frequencies that alias into a range suppressed by the digital filter may pass the analog filter. If the digital filter reaches its attenuation at f_D, then the analog filter has to suppress frequencies in the range $|n{\cdot}f_S \pm f_D|$ with n being a positive integer. For large $OSR=f_s/2f_B$ analog anti-aliasing filtering can often completely be avoided. This is shifting lowpass filtering from the analog to the digital domain. This is typical for ΔΣ ADCs, so that they can be identified by having the lowpasses ***after*** instead ***before*** the sampler.

Note: In many systems – particularly microsystems – there is hardly space for anti-aliasing filters. Techniques based on oversampling (such as ΔΣ ADCs) use high sampling rates to relax the demands of analog anti-aliasing filters or even avoid them completely.

Exercise 4: Situation sketched in Fig. 3.1.4(a): An ADC feeds a telecommunication line, required X_{dB}=56dB, Nyquist sampling, f_S=8KHz, baseband edge f_B=3.4KHz. What is the required order of the analog anti-aliasing filter?

. . . .

. . . .

. . . .

. . . .

Solution to exercise 4:

Attenuation must be replaced at $f_s/2$, therefore f_A=fs/2:

N = ceil(X_{dB}/20dB·lg(½·f_s/f_B) = ceil(*56dB*/(20dB·lg(4KHz/3.4KHZ) = ceil(39.7) = 40

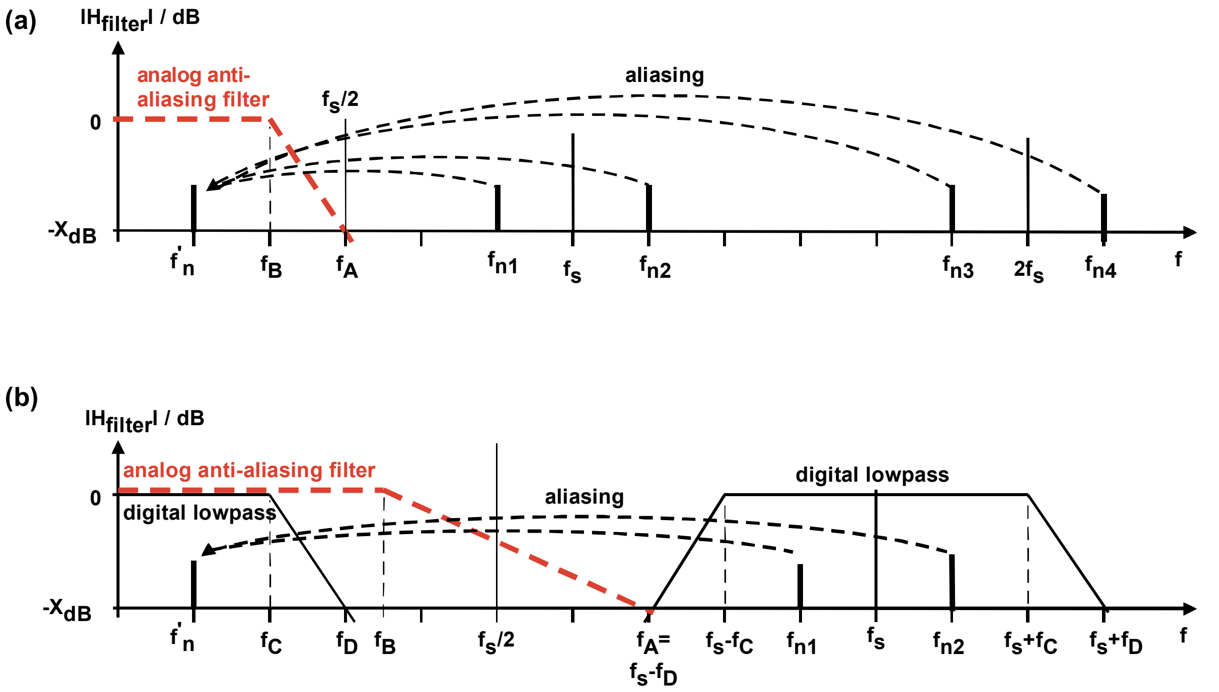


Fig. 5.4.6.4-2: Demands for an analog anti-aliasing filter: Guarantee sufficient attenuation at $f_A=f_S-f_D$ to suppress aliasing signals e.g. from f_{nx} to f'_n.

Exercise 5: Situation sketched in Fig. 5.4.6.4-2(b) above: The bandwidth available for the telecommunication customer is 3.4KHz and is achieved by a digital filter: Cutoff frequency f_C=3.4KHz, required damping $D_{dB}=X_{dB}$=89dB to be reached at f_D=4KHz, sampling frequency f_S= 500KHz. The analog anti-aliasing filter's bandwidth is set to f_B=16KHz. (It has to be > 3.4KHz but should not attenuate this frequency). What is the required order of the analog anti-aliasing filter?

Bandwidth of the analog anti-aliasing lowpass:

f_B =

. . . .

Attenuation frequency of the analog lowpass:

f_A =

. . . .

Required order of the analog anti-aliasing lowpass:

N =

. . . .

Solution to exercise 5:
Bandwidth of the analog anti-aliasing lowpass: f_B = 16KHz (given above)
Attenuation frequency of the analog lowpass: $f_A = f_S - f_D$ = (500-4)KHz = 496KHz
Required order of the analog anti-aliasing lowpass:
N = ceil(X_{dB}/(20lg(f_A/f_B)) = ceil(89dB/(20lg((500-4)KHz/16KHz)) = ceil(2,98) = 3

5.4.7 E_{clkj} : Noise Caused by Clock Jitter

If the custormer can control clock jitter – also called dither or phase noise – depends on the particular device and/or situation. (Example: *max2880*: 0.25...12.4 GHz, 0.14ps rms jitter.)
See also: https://www.maximintegrated.com/en/app-notes/index.mvp/id/3359 and

We assume a constant signal slope s'(t)= $\dot{s}$. Furthermore a Gaussian distributed sampling-timing failure τ with standard deviation σ. Then we get an error $e(\tau) = \dot{s}\tau$ with a Gaussian probability distribution having standard deviation σ. This delivers

$$w(\tau) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\tau^2}{2\sigma^2}} \quad \text{as} \quad \int_{-\infty}^{\infty} w(\tau)dt = 1 \text{ is required.}$$

The Total power is consequently given by

$$E_{clkj,rms}^2 = \int_{-\infty}^{\infty} e^2(\tau)\cdot w(\tau)d\tau = \int_{-\infty}^{\infty} (\dot{s}\tau)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\tau^2}{2\sigma^2}} d\tau = 2\int_{0}^{\infty} (\dot{s}\tau)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\tau^2}{2\sigma^2}} d\tau = \frac{2\dot{s}^2}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} \tau^2 e^{-\frac{\tau^2}{2\sigma^2}} d\tau$$

Using $\int_0^{\infty} x^2 e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{4a^3}$ for a>0 from [Bronstein-Semendjajew] with $a^2 = \frac{1}{2\sigma^2}$

delivers $E_{clkj,rms}^2 = \frac{2\dot{s}^2 a}{\sqrt{\pi}} \int_0^{\infty} \tau^2 e^{-a^2\tau^2} d\tau = \frac{2\dot{s}^2 a}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{4a^3} = \frac{\dot{s}^2}{2a^2} = (\dot{s}\sigma)^2$.

The result is surprisingly simple: $$E_{clkj,rms}^2 = (\dot{s}\sigma)^2 \quad (5.4.7\text{-}1)$$

$$E_{clkj,rms} = |\dot{s}|\sigma \quad (5.4.7\text{-}2)$$

From a very simple linear point of view this model makes sense as illustrated in Fig. 5.4.7-1. However, this is a very rough approximation and literature offers significantly more sophisticated jitter models, e.g. [1] - [4].

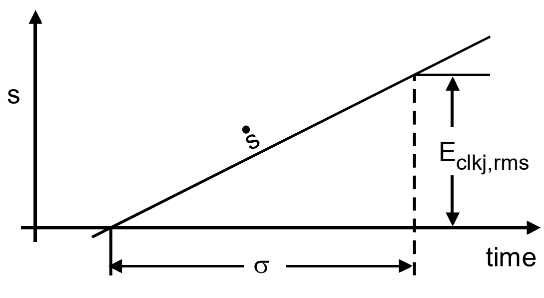


Fig. 5.4.7-1: $E_{clkj,rms} = |\dot{s}|\sigma$ as linear view.

Using (5.4.7-2) we can make different assumptions on $\dot{s}^2$. With carrier frequency ω_c and sampling time jitter σ^2 being constants we assume in Fig. 5.4.7.-2(a) that sampling the in-phase part I(t) cos(ωct) of a QAM64-signal in its extrema delivers constant signal slopes $\dot{s}^2$ from the quadrature-phase part Q(t) cos(ωct). In this case we get

$$\dot{s}^2 = (Q\omega_c)^2 \tag{5.4.7-3}$$

In the Fig part(b) we assume sampling of signal a s(t)=A sin(ωct) at random time points yielding an average signal slope at samplint time of

$$\overline{\dot{s}^2} = (A\omega_c / 2)^2 \tag{5.4.7-4}$$

Although these models might be extremely rough approximations, they give us a rough figure of what results we might expect.

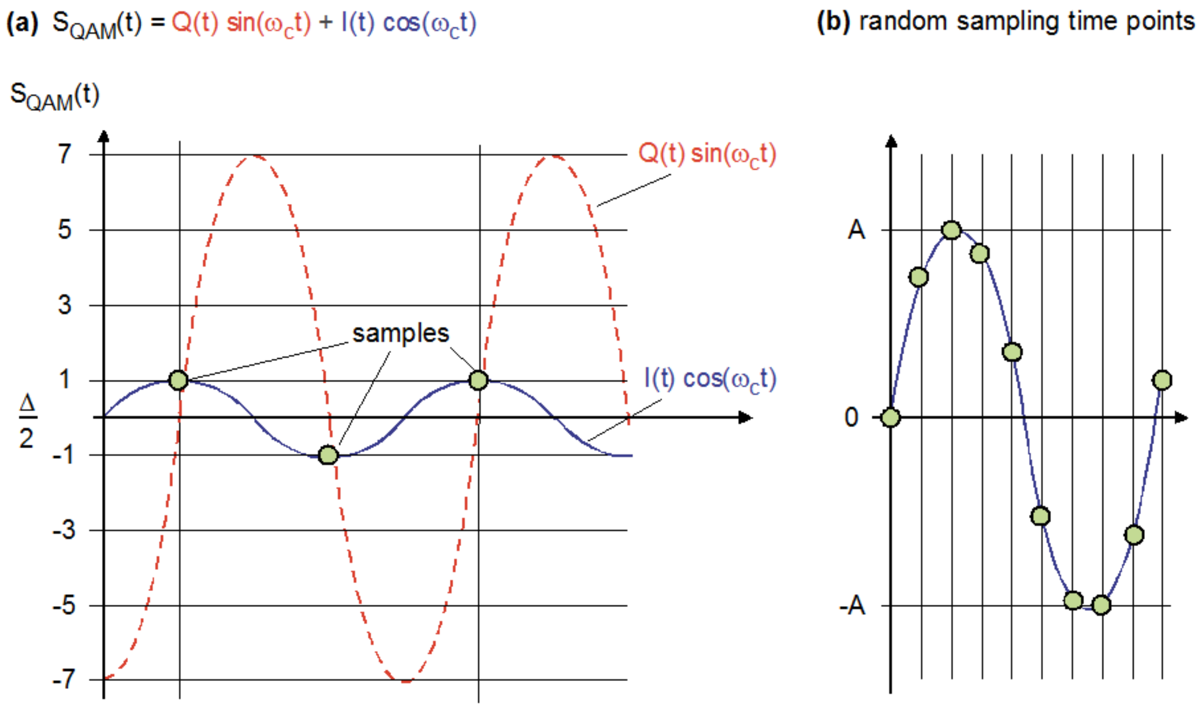


Fig. 5.4.7-2: **(a)** $|\dot{s}|$ (red) ist constant in sampling points, **(b)** We getverage $|\dot{s}|$

Bettor models are given in the references below, such as spectral noise power density

$$L(f) = \frac{\sigma_{cc}^2 f_{osc}^3}{f^2} \tag{5.4.7-5}$$

with frequency offset f from oscillator frequency f_{osc} and cycle-to-cycle jitter σ_{cc}. It is measured in dBc/Hz, with dBc being dB with respect to carrier at f_{osc}.

Some References Concerning Clock Jitter:

[1] Phase noise, Wikipedia, Available 21.06.2017: http://en.wikipedia.org/wiki/Phase_noise#Definitions.
[2] Phasenrauschen, Wikipedia, Available 21.06.2017: https://de.wikipedia.org/wiki/Phasenrauschen

[3] Maxim Integrated, Applicaton Note 3359, Clock (CLK) Jitter and Phase Noise Conversion, Available 17.06.2017: https://www.maximintegrated.com/en/app-notes/index.mvp/id/3359.

[4] Maxim Integrated, *250MHz to 12.4GHz, High-Performance, Fractional/Integer-N PLL*, 0.14 ps integrated RMS jitter, vailable 17.06.2017: https://datasheets.maximintegrated.com/en/ds/MAX2880.pdf.

[5] Analog Devices, Brad Brannon: *Sampled Systems and the Effects of Clock Phase Noise and Jitter*, Application Note AN-756, Available 21.06.2017: http://www.analog.com/media/en/technical-documentation/application-notes/AN-756.pdf.

[6] Google search: Figures about clock jitter: https://www.google.de/search?q=jitter+noise&client=firefox-b&tbm=isch&tbo=u&source=univ&sa=X&ved=0ahUKEwivjvKhoMTUAhVIZ1AKHaqpB8EQsAQIPw&biw=1645&bih=946.

Exercise 1:
We assume sampling of the sinusoidal curve $A\cdot sin(\omega_c t)$ at random time points. Given constants are standard deviation A, σ and ω_c. and V_{CC}. Compute $E_{clkj,rms}$.

Combining (5.4.7-2) $E_{clkj,rms} = |\dot{s}|\sigma$ **with (5.4.7-4)** $\overline{\dot{s}^2} = (A\omega_c/2)^2$ **delivers**
. . . .

$E_{clkj,rms}$ = σ·Aω$_c$/sqrt(2) = 1ps·(3V·2π·2.4MHz/sqrt(2) = 31.98µV
. . . .

Compute maximum SNR and SNR_{dB} achievable with $A=3V$, σ=1ps and f_c=2.4MHz, V_{CC}.=3V.

SNR = (3V)²/8 / (31.98µV)² = 1.099·10⁹ ⇔ SNR$_{dB}$ = 90.41 dB
. . . .

SNR$_{dB}$ = 10·log10(SNR) = 90.41 dB
. . . .

Exercise 2a: Fill the gaps:
A QAM64 signal is given by $S_{QAM}(t) = I(t)\cdot cos(\omega_c t) + Q(t)\cdot sin(\omega_c t)$ with
ω_c $=2\pi\cdot f_c = 2\pi\cdot 2.4GHz$ RF carrier frequency
$I(t)$ $= m\cdot\Delta/2,\ m=\pm1, \pm3, \ldots \pm M$ In-phase signal, coming as I-phase envelope,
$Q(t)$ $= n\cdot\Delta/2,\ n=\pm1, \pm3, \ldots \pm N$ Quadrature-phase signal, coming as Q-phase envelope.

It allows to represent . . . . . . different values at a time by . . . . different *I*-

and . . . . different *Q*-values: ***Q/(Δ/2), I/(Δ/2) =*** . . . .

This corresponds to . . . . parallel bits.

Exercise 2b:
A QAM signal is given by $S_{QAM}(t) = I(t)\cdot cos(\omega_c t) + Q(t)\cdot sin(\omega_c t)$ with
ω_c $=2\pi\cdot f_c = 2\pi\cdot 2.4GHz$ RF carrier frequency
$I(t)$ $= m\cdot\Delta/2,\ m=\pm1, \pm3, \ldots \pm7$ In-phase signal, coming as I-phase envelope,
$Q(t)$ $= n\cdot\Delta/2,\ n=\pm1, \pm3, \ldots \pm7$ Quadrature-phase signal, coming as Q-phase envelope.

We use a phase-locked loop (PLL) to sample in-phase signal $I(t)\cdot cos(\omega_c t)$ at its maxima. Noise ratio $NR = I_{min}/E_{clkj,rms}$ has to be at least 20dB larger than the noise power caused by the quadrature-phase signal $Q(t)\cdot sin(\omega_c t)$ at maximum amplitude $Q_{max}=7\cdot\Delta/2$. What maximum standard-deviation σ of timing failure τ can we allow for the PLL?

$I_{min}(\Delta)$ = ... , $Q_{max}(\Delta)$ = ...

NR_{dB} = 20dB corresponds to a noise ratio NR = ... in amplitude.

The worst-case slope of the Q-signal is . . . .

Its rms noise voltage due to σ is $e_{clkj,rms}$ = ...

Use this is $e_{clkj,rms}$ to compute the σ we can allow for sampling the Q-signal

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Solutions:

Exercise 1:
We assume sampling of the sinusoidal curve $A\cdot sin(\omega_c t)$ at random time points. Given constants are standard deviation A, σ and ω_c. and V_{CC}. Compute $E_{clkj,rms}$.

Combining (5.4.7-2) $E_{clkj,rms} = |\dot{s}|\sigma$ with (5.4.7-4) $\overline{\dot{s}^2} = (A\omega_c/2)^2$ delivers

$E_{clkj,rms}$ = σ·Aω$_c$/sqrt(2) = 1ps·(3V·2π·2.4MHz/sqrt(2) = 31.98µV

Compute maximum SNR and SNR_{dB} achievable with $A=3V$, σ=1ps and f_c=2.4MHz, V_{CC}.=3V.

SNR = (3V)^2/8 / (31.98µV)^2 = 1.099·10^9 ⇔ SNR$_{dB}$ = 90.41 dB

Exercise 2a: Fill the gaps:
It allows to represent . . 64 . . different values at a time by . . . 8 . . . different I-
and . . . 8 . . . different Q-values: ***Q/(Δ/2), I/(Δ/2)*** = ±1, ±3, ± 5, ± 7 .
This corresponds to . . . . ld(64) = ln(64)/ln(2) = 6 . . . . . parallel bits.

Exercise 2b:
$I_{min}(\Delta)$ = ... Δ/2 . . . . , $Q_{max}(\Delta)$ = ... Δ·7/2 . . . . . .
NR_{dB} = 20dB corresponds to a factor NR = . . 10 . . . in amplitude.
The worst-case slope of the Q-signal is . . s' = Q_{max} 2π f_C = 7(Δ/2) 2π f_C.
Its rms noise voltage due to σ is $e_{clkj,rms}$ = s'σ =.Q_{max} 2π f_c = 14πσ (Δ/2)·f_c
Use this is $e_{clkj,rms}$ to compute the σ we can allow for sampling the Q-signal

From NR ≤ I_{min} / $e_{clkj,rms}$ = I_{min} / (s'σ)
= (Δ/2) / (σ 2π 7(Δ/2) f_c)
= 1 / (σ 14π f_c)

follows: σ ≤ 1 / (NR 14π f_c) = 1/(10·14π·2.4GHz) = 0.947ps
Note: this is 0.227% of a 2.4GHz period, which is 1/2.4GHz = 417ps

Sources of clock jitter are particularly circuits like

1. DLL: Delay locked loop
2. PLL: Phase-locked Loop
3. CDR: Clock-Data Recovery Circuit
4. Software: Clocks signals computed by software

1. DLL: Delay Locked Loop
Operates a (typically voltage) controlled delay. It can delay a clock signal so that a retardation (for example caused by buffering the signal) can be compensated for.
+ ***Best (=smallest) figures of phase noise.***
- Frequency differences cannot be compensated for (use signals from same clock source!).

2. PLL: Phase Locked Loop
Operates a (typically voltage) controlled local oscillator (LO). It can shift frequencies to match received frequency and phase. It is used e.g. for demodulation of FM and AM radio signals.
+ Can synchronize its local oscillator (LO) to a range of external frequencies
+ Better phase noise than CDR, worse than DLL
- Continuous, uniform oscillation required, no "missing bits" on the data stream!

3. CDR: Clock Data Recovery Circuit
operates a (typically voltage) controlled local oscillator (LO) with a phase detector, that can swallow missing bits on a data stream. Used to recover the clock signal for USB bit-streams: However, in the USB community the CDR is mostly termed PLL. A good CDR can hold synchronicity over some 1000 bits without an edge (i.e. some 1000 ones ore zeros only)
+ Can synchronize its local oscillator (LO) to a data stream with randomly arriving bits.
± Better phase noise than software, worse than DLL and PLL.

3. Software generated clock signals
The author's experience with 1's and 0's set be software to generate a clock signal are bad. The process of software processing and interrupt handling within a CPU is difficult to control and phase noise is quite unacceptable.
- No special hardware (DLL, PLL, CDR) required (making it attractive to many engineers).
- Typically poor phase noise.

5.4.8 $E_{T\&H}$: Noise Caused by the Track & Hold Circuit

5.4.8.1 Ideal Sample & Hold Process

Using the fact that $\int_{-\infty}^{\infty}\delta(t)dt = 1$ the process of taking a single sample is mathematically modeled as

$$y(a) = \int_{-\infty}^{\infty} y(t)\delta(t-a)dt\,.$$

with δ(t) being the Dirac function. Sampling, i.e. the process of translating a time-continuous to a time-discrete function, is described as

$$y(n) = \sum_{n} \int_{-\infty}^{\infty} y(t)\delta(t-nT)dt$$

Unfortunately, there is no technical realization of this mathematical concept known to the author. In real systems track and hold circuits are used.

5.4.8.2 Track & Hold Process Assuming a Maximum Voltage Step

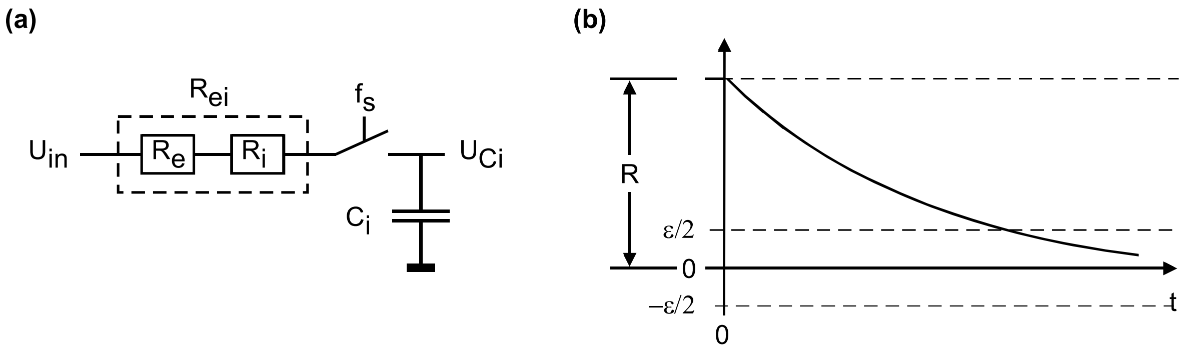


Fig. 5.4.8.2: (a) Track & Hold Circuit, **(b)** waveform on the holding capacitor.

A typical track & hold circuit can be modeled as a switch with *RC* lowpass as shown in the Fig. Above. The resistor R_{ei} consists of an internal resistor R_i and an external resistor R_e, which is the output impedance of the signal source:

$R_{ei} = R_i + R_e$

The customer's impact on this system is given by R_e and an the track- & hold-times of the sampler. During tracking, the switch is conducting and during hold it is open. In the worst case, a maximum initial voltage U_{Ci0} (indicated as range *R* in the graphics) on the capacitor has to be discharged to zero. The discharge curve of the capacitor's voltage is then

$$U_{Ci}(t) = U_{Ci0} \cdot e^{-\frac{t}{R_{ei}C_i}} \quad \Leftrightarrow \quad t = R_{ei}C_i \cdot \ln \frac{U_{Ci0}}{U_{Ci}(t)}$$

reaching the final accuracy of $|U_{Ci}(t_{Track})| \leq \Delta/2^k$ with settling time

$$t_{Track} \geq R_{ei}C_i \ln \frac{U_{Ci0}}{\Delta / 2^k} .$$

For an NoB-bit ADC with $\Delta = U_{Ci0}/2^{NoB}$ we get

$$t_{Track} = R_{ei}C_i \cdot \ln\left(\frac{U_{Ci0}}{\Delta / 2^k}\right) = R_{ei}C_i \cdot \ln\left(\frac{U_{Ci0}}{U_{Ci0} / 2^{NoB+k}}\right) = R_{ei}C_i \cdot \ln\left(2^{NOB+k}\right).$$

Using $ln(x^n) = n \cdot ln(x)$ delivers the formulae typically found in data sheets for NoB-bit ADCs:

$$\boxed{t_{Track} = R_{ei}C_i (NoB + k)\ln 2 \cong R_{ei}C_i (NoB + k) \cdot 0.693}.$$

With sampler cut-off frequency $f_C = 1/(2\pi R_{ei}C_i)$ this translates to

$$\boxed{t_{Track} = \frac{1}{2\pi f_C}(NoB + k)\ln 2 \cong \frac{0.11 \cdot (NoB + k)}{f_C}}.$$

Typically k=1 is assume and consequently an accuracy of $\Delta/2$ to be achieved.

According to Nyquist the maximum bandwidth that can be sampled is

$$f_B = \frac{1}{2} f_S = \frac{1}{2} \frac{1}{t_{Track} + t_{Hold}}$$

Exercises

Exercise 1:

Assume t_{Track}=9ns sampler-settling time and an ADC's conversion time of t_{Hold}=11ns. What is the maximum possible sampling frequency f_S of the sampling system? (formula + value)

. . . .

. . . .

Exercise 2:

Regard your sampler as *RC* lowpass composed of $R_{ei}=1K\Omega$, C_i=1pF and a required accuracy *NoB*=10 bits. Compute the required minimum time for t_{Track}.

. . . .

. . . .

Exercise 3:

Compute the bandwidth, f_B, when $t_{Track}=t_{Hold}$ for the setup in exercise 2.

. . . .

. . . .

Exercise 4:

Compute the cutoff frequency for the setup in exercise 2.

. . . .

. . . .

Solutions

Solution to exercise 1:

Assume t_{Track}=9ns sampler-settling time and an ADC's conversion time of t_{Hold}=11ns. What is the maximum possible sampling frequency f_S of the sampling system? (formula + value)

$f_S = 1/T_S = 1/(t_{Track}+t_{Hold}) = 1/(9ns+11ns) = 1/20ns = 50\ MHz$

Solution to exercise 2:

Regard your sampler as *RC* lowpass with R_{ei}=1KΩ, C_i=1pF, *NOB*=10. Compute the required minimum time for t_{Track} with k=1.

$t_{Track} \geq (NOB+k)\ R_{ei}C_i \cdot \ln(2) = (10+1)\ 10^3\Omega\cdot10^{-12}F \cdot 0{,}693 = 7.62\ ns$

Solution to exercise 3:

Compute the bandwidth, f_B, that can be sampled when $t_{Hold} = t_{Track}$ for the setup in exercise 2.

$f_B(t_{Hold}=t_{Track}) = 0.5\cdot f_S = 0.5\ /\ (2\cdot7.62ns) = 32.8\ MHz$

Solution to exercise 4:

Compute the cutoff frequency of the sampler's *RC* lowpass for the setup in exercise 2.

$f_C = 1/(2\pi R_{ei}C_i) = 1/(2\pi\cdot10^3\Omega\cdot10^{-12}F) = 159\ MHz$

5.4.8.3 Track & Hold Process Applied on Dynamic Input

The following considerations for dynamic input are irrelevant for the ADA exam.

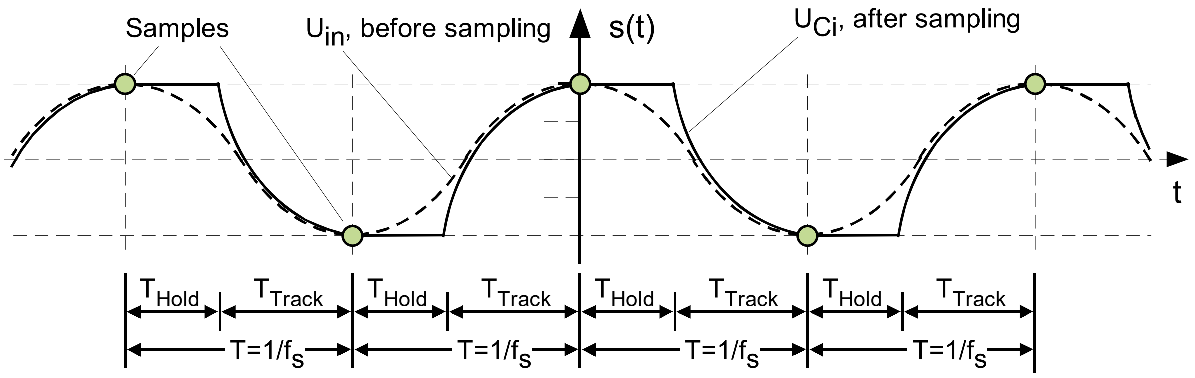


Fig. 5.4.8.3: (a) sampling system, **(b)** RC discharge curve

The formula found in data sheets and text books is typically $t_{Track}(NOB) = (NOB+1)R_{ei}C\ ln2$. The consideration below sets some question marks behind it.

Assuming the case of ideal sampling with t_{Hold}=0. Then $U_{Ci}(f) = H_{LP}(f)\cdot U_{in}(f)$ with

$$H_{LP}(f) = \frac{1}{1 + j\dfrac{f}{f_C}}$$

and $f_C = \dfrac{1}{2\pi R_{ei} C_i}$. The sampler's amplitude attenuation of a sinusoidal signal is consequently

$$\left|\frac{U_{Ci}(f)}{U_{in}(f)}\right| = |H_{LP}(f)| = \frac{1}{\sqrt{1+\left(\dfrac{f}{f_C}\right)^2}}$$

Let signal $s(t) = (R/2)\sin(2\pi f_B t)$ at bandwidth edge f_B span signal range R, witch is subdivided by a NOB-bit ADC into 2^{NOB}-1 deltas according to $\Delta=R/(2^{NOB}-1)\approx R\cdot 2^{-NOB}$. The maximum amplitude error caused by the sampler's attenuation is

$$E_{Track} = U_{in,\max} - U_{Ci,\max} = \frac{R}{2} - |H_{LP}(f_B)|\frac{R}{2} = \frac{R}{2} - \frac{1}{\sqrt{1+\left(\dfrac{f_B}{f_C}\right)^2}}\cdot\frac{R}{2}$$

On the other hand we have

$$\frac{\Delta}{2} = \frac{1}{2}\cdot\frac{R}{2^{NOB}-1} \cong \frac{R}{2^{NOB+1}}$$

From $|E_{Track}| \le \frac{\Delta}{2}$ we get

$$\frac{R}{2} - \frac{R/2}{\sqrt{1+\left(\frac{f_B}{f_C}\right)^2}} \le \frac{R}{2^{NOB+1}} \quad => \quad 1 - \frac{1}{\sqrt{1+\left(\frac{f_B}{f_C}\right)^2}} \le 2^{-NOB} \quad => \quad 1+\left(\frac{f_B}{f_C}\right)^2 \le \frac{1}{\left(1-2^{-NOB}\right)^2}$$

and consequently

$$\frac{f_B}{f_C} \le \sqrt{\left(\frac{1}{1-2^{-NOB}}\right)^2 - 1}$$

For x<<1 we can use $\frac{1}{1-x} \cong 1+x$ and $(1+x)^2 \cong 1+2x$. Substituting $x=2^{-NOB}$ yields

$$\frac{f_B}{f_C} \le \sqrt{\left(\frac{1}{1-2^{-NOB}}\right)^2 - 1} \approx \sqrt{\left(1+2^{-NOB}\right)^2 - 1} \approx \sqrt{1+2^{-NOB+1} - 1} = \sqrt{2^{-NOB+1}} = 2^{-(NOB-1)/2}$$

In summary, to prevent the sampler's RC lowpass from causing amplitude errors > Δ/2 the bandwidth of the sampled signal has to be limited to

$$\boxed{\frac{f_B}{f_C} \le \frac{1}{2^{(NOB-1)/2}} = 2^{-\frac{NOB-1}{2}}}$$

Exercise 6:
Compute the theoretical maximum of bandwidth, f_B, for the sampler and ADC in exercise 2 (having f_C=159.15 MHz from $R_{ei}=1K\Omega$, $C_i=1pF$, $NOB=10$).

. . . .

Exercise 7:
Compute the transfer function $H_{LP}(f_B)$ of the sampler's RC lowpass and show that the error is ca. Δ/2.

. . . .

Solution to exercise 6:
Compute the bandwidth, f_B, when $t_{Track}=t_{Hold}$ for the sampler and ADC in exercise 2.
$f_B = f_C/2^{(NOB-1)}/2 = f_C/2^{(10-1)/2} = 159.15\ MHz/2^{4.5} = 159.15\ MHz/22.6 = 7.03\ MHz$

Solution to exercise 7:
Compute the transfer function $H_{LP}(f_B)$ of the sampler's RC lowpass and show that the error is ca. Δ/2.
$H_{LP}(f_B)$ = 1/sqrt(1+$(f_B/f_C)^2$) = 1/sqrt(1+$(7.03/159)^2$) = 0.9990, so that 1-$H_{LP}(f_B)=10^{-3}$. Considering a signal range of ±R/2 subdivided into $2^{10}\Delta \approx 1000\Delta$, 1/1000 of R/2 corresponds to a half Δ.

5.4.9 E_{otex} : Other External Noise Sources

There are several other external noises sources as Johnson noise of resistors or 1/f (pink) noise of FETs etc. Noise models are typically difficult to obtain. Here we consider thermal or so-called Johnson noise, which has an easy and reliable model, as well as pink noise.

5.4.9.1 E_J : *Johnson* = Thermal Noise

Due to temperature, atoms oscillate around their atomic lattice sites kicking electrons around which can be measured as thermal noise.

While capacitors and inductors do not contribute Johnson noise, any resistor has a noise spectral density of

$P_J'(f,T) = 4kT$ in $J = VAs = Ws = W/Hz$

with Boltzmann's constant k=1.38065·10^{-23}J/K. and T being the absolute temperature in Kelvin (=temperature in °C+273.15). Note that the physical dimension of noise power density is ***power/Hz***! This density is constant over the frequency axis. (In reality, this would deliver an infinite power for infinity bandwidth, but this formula is valid up to the Terra-Hertz range.)

Examples:

$$P_J'(f,T_1 = 300K) = 4 \cdot 1,380662 \cdot 10^{-23}(VAs/K) \cdot 300K = 1,6568 \cdot 10^{-20} VAs$$

$$P_J'(f,T_2 = 600K) = 3,3136 \cdot 10^{-20} VAs$$

$$P_J'(f,T_3 = 900K) = 4,9704 \cdot 10^{-20} VAs$$

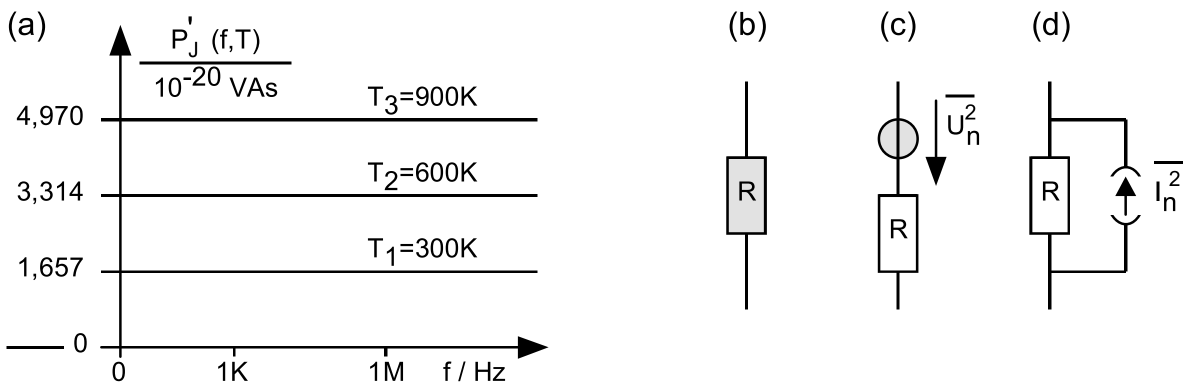


Fig. 5.4.9.1.1: (a) Noise power density of a resistor for 3 temperatures, **(b)** noisy resistor, **(c)** equivalent circuit with noise voltage and **(d)** noise current source.

Example: The thermal noise power generated by a resistor in frequency band B=10...11 KHz at a temperature of T_1=300K is

$$P_J = \int_{10KHz}^{11KHz} P_J^{'}(f, T_1 = 300K) \cdot df = P_J^{'} \cdot B = 4kTB = 1{,}657 \cdot 10^{-20} VAs \cdot 1000 Hz = 1{,}657 \cdot 10^{-17} VA$$

As *Johnson* noise is constant („white“) over frequency, integration reduces to a simple multiplication with bandwidth B:

$$\boxed{P_J = 4kTB}$$

As $P_J = u_{J,rms}^2 / R = i_{J,rms}^2 \cdot R$ this power is measurable as

noise voltage $u_{J,rms} = \sqrt{P_J \cdot R} = \sqrt{4kTBR}$ in V $\Leftrightarrow$ $u_{J,rms}^{'} = \sqrt{P_J^{'} \cdot R} = \sqrt{4kTR}$ in $V/\sqrt{Hz}$

noise current $i_{J,rms} = \sqrt{P_J / R} = \sqrt{4kTB/R}$ in A $\Leftrightarrow$ $i_{J,rms}^{'} = \sqrt{P_J^{'} / R} = \sqrt{4kT/R}$ in $A/\sqrt{Hz}$

Consequently, in our converter noise models with error $E_{J,rms}$ being a voltage we get

$$\boxed{E_{J,rms}^2 = 4kTBR \text{ in } \mathrm{V}^2 \quad \Leftrightarrow \quad E_{J,rms} = \sqrt{4kTBR} \text{ in } \mathrm{V}}$$

See also: “Tontechnik-Rechner – segpielaudio”, available: http://www.sengpielaudio.com/calculator-noise.htm.

Exercise 1:

In a design with signal range of 0...V_{CC}=3.3V you have a thermal noise power budget corresponding to an accuracy of 14 bit. Your Bandwidth is B=100MHz. Maximum operating temperature is T = 400K. What is the maximum resistor allowed at the ADC’s input? (k=1.38·10^{-23}J/K) (Solution at → next page)

Exercise 2:

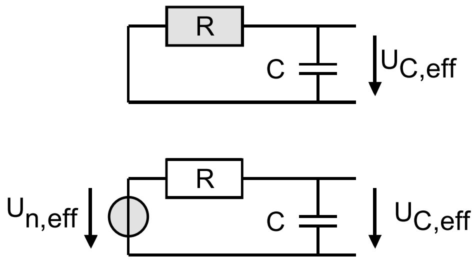
Same as exercise 1 with B=2.4GHz. Maximum resistor R=? (Solution at → next page)

Exercise 3: Compute rms thermal noise density across capacitor C

Fig. 5.4.91.2:

(a) RC lowpass with noisy resistor.

(b) RC lowpass like above with noiseless resistor and equivalent noises source $u'_{R,rms}$.



Compute the noise power across capacitor C in Fig. 5.4.9.1.2 caused by resistor *R*. Capacitors and inductors do not generate thermal noise.

Compute the spectral thermal noise power density $u'^2_{R,rms}(f)$ generated by resistor *R* as a function of *k, T, R* with *k* being Boltzmann's constant and *T* absolute temperature in K.

$$u'^2_{R,rms}(f) \quad = \quad 4kTR$$

Let $H_{LP}(f)$ be the transfer function of the low-pass. What is the spectral noise density across C as a function of $u'_{R,rms}(f)$ and $H_{LP}(f)$?

$$u'^2_{C,rms}(f) \quad = \quad u'^2_{R,rms} \left| H_{LP}(f) \right|^2$$

For a first order low-pass with pole in f_B the transfer function is $H_{LP}(f)=1/(1+jf/f_B)$. What is the spectral noise density across C as a function of $u'_{R,rms}(f)$ and f/f_B?

$$u'^2_{C,rms} \quad = \quad u'^2_{R,rms} \frac{1}{1+(f/f_B)^2}$$

Exercise 4: Approximation of $u^2_{C,rms}$.

We approximate | $H_{LP}(f)$| piecewise with its asymptotes:

$$\left| H_{LP}(f) \right| \cong \left| H_{LP,approx}(f) \right| = \begin{cases} 1 & if \quad f \le f_B \\ f_B / f & if \quad f \ge f_B \end{cases}$$

Compute $u^2_{C,rms}$ by piecewise integration as function of $u'_{R,rms}$ and f_B, both being constants.

$$u^2_{C,rms} \cong \int_{f=0}^{\infty} u'^2_{R,rms} \left| H_{LP,approx}(f) \right|^2 \cdot df \quad = \quad u'^2_{R,rms} \left[\int_{f=0}^{f_B} 1^2 df + \int_{f=f_B}^{\infty} \left(\frac{f_B}{f} \right)^2 df \right] = 2 f_B$$

Exercise 5: Exact computation of $u^2_{C,rms}$.

As the lowpass is of first order, we can calculate an accurate solution of the integral using the mathematical textbook equation $\int \frac{df}{1+(x/a)^2} = a\arctan(x/a)$.

$$u^2_{C,rms} \cong \int_{f=0}^{\infty} u'^2_{R,rms} \left|H_{LP}(f)\right|^2 \cdot df = u'^2_{R,rms} \int_{f=f_B}^{\infty} \frac{df}{1+(f/f_B)^2} = f_B\left[\arctan(f/f_B)\right]_{f=0}^{\infty} = f_B\frac{\pi}{2} \cong 1.57 f_B$$

. . . .

. . . .

Solution to exercise 1:

Planning a design with signal range of 0...V_{CC}=3.3V you have a thermal noise power budget corresponding to an accuracy of 14 bit. Your Bandwidth is B_1=100MHz. maximum operating temperature is T=400K. What is the maximum resistor allowed at the ADC's input? (k=1.38·10^{-23}J/K)

Signal power is $U^2_{rms,\max} = \frac{V^2_{CC}}{8}$, available power budget $\left(\frac{U_{rms,\max}}{2^{14}}\right)^2 = \frac{V^2_{CC}/8}{2^{2*14}}$ =5.071 10^{-9}V^2

Consequently: $4kTBR = \frac{V^2_{CC}/8}{2^{2*14}}$ => $R = \frac{V^2_{CC}/8}{4kTB \cdot 2^{2*14}}$ = 2,30 KΩ

Solution to exercise 2:

Same as exercise 1 with a bandwidth of B_2=2.4GHz. Maximum resistor R=?

We compensate for the division by B=100MHz by a corresponding multiplication with B and then divide by the new bandwidth B_2=2.4GHz: **2.30KΩ·B/B₂ = ·2.30KΩ·100MHz/2.4GHz = 95.65 Ω**

Solution to exercise 3:

The integration $\int_{f=0}^{\infty} \left|H_{approx}(f)\right|^2 df$ delivers $2f_B$. In the exact computation we get

$$\int_{f=0}^{\infty} \left|H_{exact}(f)\right|^2 df = \int_{f=0}^{\infty} \frac{df}{1+(f/f_B)^2} = f_B\left[\arctan(f/f_B)\right]_{f=0}^{\infty} = f_B\left[\frac{\pi}{2} - 0\right] = f_B\pi/2 .$$

Consequently, the exact result here is obtained from the approximated result with

$$u_{C,rms,exact} = u_{C,rms,approx}\sqrt{\frac{\int_{f=0}^{\infty} w_{exact}df}{\int_{f=0}^{\infty} w_{approx}df}} = u_{C,rms,approx}\sqrt{\frac{f_B\pi/2}{2f_B}} = u_{C,rms,approx}\frac{\sqrt{\pi}}{2} = 510.2nV$$

5.4.9.2 E_{pink} : 1/*f* = Pink = Flicker Noise

Particularly in semiconductors and semiconductor/oxide interfaces, we find the so-called flicker noise, also termed $1/f$ noise or pink noise. "Pink" stems from the fact that $1/f$-shaped visible light would be perceived pink. Fig. 5.4.9.2 illustrates a typical $1/f$ spectral noise density, part (a) with linear and (b) with logarithmic scaling. Note that in Fig. part (b) we have a slope of -10dB/dec (not -20dB/dec!), as we plot power (not amplitude) versus frequency. Quantitatively pink noise depends on the device.

The simplest mathematical model for pink noise requires two parameters:

- P'_{NF}: the noise floor's spectral power density, and
- f_{NC}: the noise corner frequency where pink noise equals noise floor power density.

For typical operational amplifiers f_{NC} is some 100Hz. For typical MOSFETS pink noise becomes dominant over thermal noise below 100Hz [Hau99].

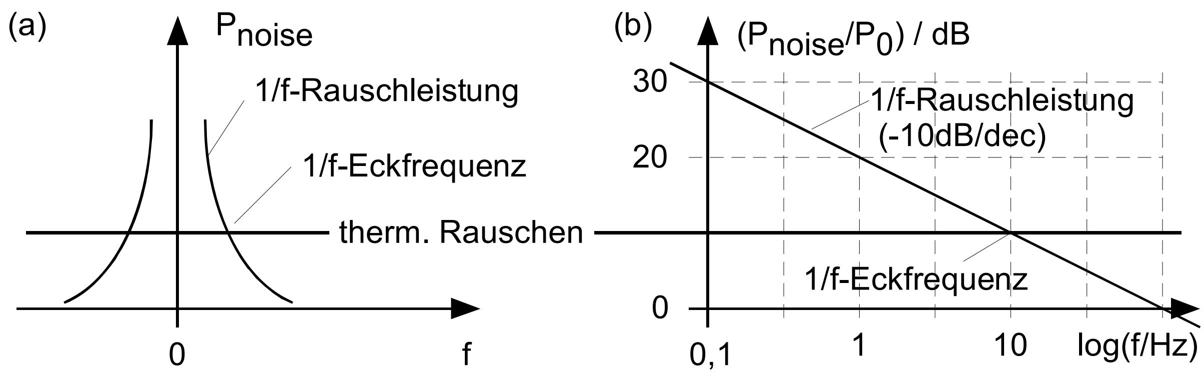


Fig. 5.4.9.2: 1/f noise with **(a)** linear and **(b)** logarithmic scaling.

Modeling

$P'_{pink}(f) = P'_{NF} \cdot f_{NC}/f$.

A possible noise floor related to resistors was

$P'_{NF} = 4kT.$

The total pink noise-power in frequency band $f_1 \ldots f_2$ becomes

$$P_{pink}(f_1, f_2) = P'_{NF} \int_{f_1}^{f_2} \frac{f_{NC}}{f} df = P'_{NF} f_{NC} \ln \frac{f_2}{f_1} \; .$$

Offset.
The offset drift e.g. of operational amplifiers versus time and temperature can be seen as low frequency $1/f$ noise. Offset at frequency 0Hz is theoretically infinite in the $1/f$ model but practically impossible, as it corresponds to an infinitely long period of time.

Noise references cited

[Hau99] Hausherr, B., "Flicker-Rauschen: Eigenschaften und Simulation", Elektronik, Heft 7, p. 64-69, 7. April 1999.

[Böd07] Bödiger, Wolfgang, "Rauschen ausgeblendet – Genaue OPVs durch Autozero-Technik“, Design & Elektronik, Heft 09, September 2007, pp. 18-20.

5.4.9.3 E_{cur} : Current Noise

Hold a needle into a smooth jet of water from a garden hose and observe the effect. The small needle will strongly disrupt the water jet. Then try the same with a comb or a brush, they will destroy the smooth water jet. The perturbations observed may give you a figure of how charged doping atoms or grainy material disturbs a smooth current flow. For this reason, metal film resistors cause less current noise than grainy carbon layer resistors, and poly crystalline silicon causes more current noise than mono crystalline silicon.

Current noise models are strongly material dependent and are in many cases difficult to obtain.